Neutrinos and Earth-Sized Quantum Devices

Matheus Hostert

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Resources

Every particle physicist needs a reference to remember all the masses, lifetimes, and properties of the zoo of particles we know. Fortunately, the links below contain all the information you can possibly need (for free!). You can also order your own booklet if you'd like a physical copy (but we better spare the trees if we can!)

- `pdglive.lbl.gov/`
- The Particle Data Group has several review pages on particle physics and cosmology: `https://pdg.lbl.gov/2021/reviews/contents_sports.html`.

1 First session

1) Natural units Particle physicists like to express physical quantities in units that are *not* defined by human standards, like meters and seconds, but rather by Nature itself. As you learned in your Special Relativity course, the speed of light in a vacuum,

$$c = 299\,792\,458 \text{ m/s}, \tag{1.1}$$

is a constant for all observers; therefore, it is as universal a quantity as one can get. Another universal constant is the Planck constant,

$$\hbar = \frac{h}{2\pi} = 1.054571817 \times 10^{-34} \text{ J.s} = 6.582119569 \times 10^{-16} \text{ eV.s}, \tag{1.2}$$

where we just used the definition of an electron-Volt in the second equality. Instead of expressing them in terms of the SI units, we can instead use them as our "standard rulers". This trick is the essence of what is called *working in natural units*.

The simplest convention we can pick is to say that $c = \hbar = 1$. This assignment relates distances, time intervals, and energies to each other in terms of fundamental constants. An immediate result is that in natural units, one second is simply

$$1 \text{ s} = 299\,792\,458 \text{ m}, \tag{1.3}$$

allowing us to translate between time intervals and distances using the speed of light. In addition, instead of talking about speeds and velocities in units of meters per second, we can now simply state them as fractions of the speed of light. For example, if your typical walking speed is $v = 1.4$ m/s, then in natural units we say that you walk at speed of $v = 4.6 \times 10^{-9}$, that is, at a fraction of $v = 4.6 \times 10^{-9}$ of the speed of light. Note that there are no dimensions in this number anymore since they are always interpreted as fractions of the speed of light, which is just unity ($c = 1$).

This makes physical theories easier to write down and often easier to interpret. For example, the famous relativistic energy equation for a particle with mass m and momentum p

$$E^2 = m^2c^4 + p^2c^2, \tag{1.4}$$

in natural units becomes,

$$E^2 = m^2 + p^2. \tag{1.5}$$

The equivalence between mass, momentum and energy becomes much more immediate in this way since E, m, and p must now all have the same units. By convention, they are all expressed in electron-Volts (eV) – see exercise a) below.

One more example is the De Broglie wavelength of a photon. We know that

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar c}{E} \to \lambda = \frac{2\pi}{E} \text{ (in natural units).} \tag{1.6}$$

So, in natural units, the inverse-proportionality of length and energy becomes immediate, and lengths are expressed in terms of inverse-electron-Volts (1/eV). In fact, this relationship means all lengths and time intervals can be interpreted as inverse energy. We do this because the additional factors of c and $\hbar$ in our theories are often superfluous and not very insightful. It also helps to reduce the number of units to keep track of – at the end of the day, most quantities of interest are expressed in terms of energy. We will practice with some examples below:

Question a) The mass of a proton in Kilograms is approximately $m_p \simeq 1.67 \times 10^{-27}$ kg. Using natural units, find the mass of a proton at rest in GeV.

This one we did together in today's session, but I go through it again in a different way below.
We first find one kg in terms of the other units. We can make use of the Planck constant in units of J.s = (kg m$^2/s^2$).s and write

$$\hbar = 1 \implies \text{kg} \simeq 10^{34}\,\frac{\text{s}}{\text{m}^2} \simeq 3 \times 10^{42}\,\frac{1}{\text{m}}, \tag{1.7}$$

where we multiplied by $c = 1$ in the last step. One useful quantity (which is also unity) is $\hbar c \simeq 197$ MeV.fm, where fm $= 10^{-15}$ m. Multiplying Equation (1.7) by $\hbar c$, we get

$$\text{kg} \simeq 6 \times 10^{29} \text{ MeV}, \tag{1.8}$$

and finally,

$$m_p \simeq 1000 \text{ MeV} = 1 \text{ GeV}, \tag{1.9}$$

which is pretty close to the desired value of $m_p = 938.272$ MeV.

Question b) Today, typical collisions at the Large Hadron Collider (LHC) have a total energy of $E = 13$ TeV. What are the typical distances probed by these experiments? You can think of collisions at the LHC as the exchange of particles of light carrying energies of $E = 13$ TeV between two protons.

Again, using $\hbar c \simeq 197$ MeV.fm, we can calculate the De Broglie wavelength of photons produced at the LHC by using the equation Equation (1.6) and multiplying both sides by $\hbar c$,

$$\begin{aligned} \lambda_{\text{LHC}} = \lambda_{\text{LHC}} \times \hbar c &= \frac{2\pi}{13 \text{ TeV}} \times \hbar c \qquad (1.10) \\ &= \frac{2\pi}{13 \times 10^6 \text{ MeV}} \times 197 \text{ MeV.fm} \simeq 9.5 \times 10^{-5} \text{ fm} \simeq 9.5 \times 10^{-20} \text{ m.} \end{aligned}$$

Compare these distances with the De Broglie wavelength of a proton at rest (the right way to think about is to imagine a photon with the same energy as the proton's rest energy, i.e. $E = m_p$. This photon is then said to have a "wavelength of the size of a proton".)

$$\lambda_p = \frac{2\pi}{m_p} \simeq 1.3 \times 10^{-15} \text{ m} \simeq 1 \text{ fm.} \qquad (1.11)$$

Question c) The W boson has a mass of $M_W \simeq 80$ GeV. Even though it is much heavier than the muon, $m_\mu \simeq 105$ MeV, it is still the mediator particle responsible for muon decay. i) Sketch the Feynman diagram responsible for this process (Hint: muons decay predominantly into two neutrinos and an electron).

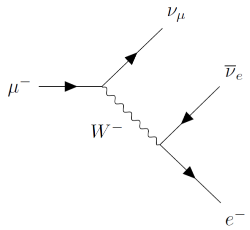
ii) In the Standard Model, the lifetime of the muon can be calculated as

$$\frac{1}{\tau_\mu} \simeq \frac{G_F^2 m_\mu^5}{192\pi^3}, \qquad (1.12)$$

where G_F is called Fermi's constant. How long does a muon live in its rest frame?

iii) Compare your answer with the lifetime of a rho meson, $\rho(776)$. You can find the rho meson lifetime on the PDG website under "Mesons", and "Light unflavored." Why are they so different? (Hint: the rho meson is so short-lived that its lifetime is given in terms of its width Γ_ρ, and not of its actual lifetime τ_ρ. In natural units, you can convert one into the other by using $\Gamma_\rho = 1/\tau_\rho$.)

i) With time flowing to the right, we have



ii) Fermi's constant determines the strength of weak interactions and is given by $G_F = 1.1663787 \times 10^{-5}$ GeV^{-2}. The fact that the dimensionless quantity $G_F \times m_\mu^2 \simeq 10^{-7}$ is small is the reason why weak interactions are so weak. Let's compute the muon width first $(\Gamma = 1/\tau)$[a],

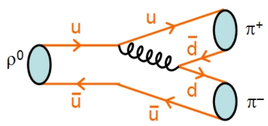
$$\frac{1}{\tau_\mu} \simeq \left(10^{-5}\ \text{GeV}^2\right)^2 \times (0.1\ \text{GeV})^5 \frac{1}{192\pi^3} \simeq \times 10^{-19}\ \text{GeV}, \tag{1.13}$$

which can be translated into seconds using $\hbar$,

$$\tau_\mu \simeq 3.09 \times 10^{-6}\ \text{s}. \tag{1.14}$$

This may seem like a very short time, but in the context of particle physics, it is one of the longest-lived unstable particles we know of.

iii) For comparison, the $\rho(776)$ vector meson decays predominantly via the strong force; see the diagram below.



The neutral and charge ρ mesons have very similar widths, $\Gamma \simeq 150$ MeV, which translated into a lifetime of

$$\tau_\rho \simeq 4 \times 10^{-24}\ \ \text{s}, \tag{1.15}$$

which is several many orders of magnitude smaller. This decay is so fast precisely because of the strength of the strong force.

[a] The width of a particle is the intrinsic uncertainty in its rest mass, which is why they decay. The larger the width of a particle (in eV, for example), the shorter-lived it is. One way to understand this is via the uncertainty principle, $\Delta t \Delta E \gtrsim \hbar/2$. The larger the uncertainty on the particle's rest energy (ΔE, in this analogy), the shorter its lifetime (Δt).

Bonus Question 1) Neutrino Delay In 1987, several neutrino experiments worldwide observed neutrinos from a supernova called SN1987A. This supernova was a star that exploded at a distance of $D = 52$ kpc from the Earth, emitting a huge number of neutrinos in the process. One experiment observed different neutrinos with different energies in the period of 9 s. The first neutrino was detected at time $t_1 = 0$ s with energy $E_{\nu_1} = 20$ MeV, a second one at time $t_2 = 0.3$ s with energy $E_{\nu_2} = 0.3$ MeV, and one last one at time $t_3 = 9$ s and $E_{\nu_3} = 10$ MeV.

Assuming that the supernova burst was instantaneous (all neutrinos were emitted simultaneously at $t < 0$), and based on these experimental measurements, is this data consistent with neutrinos being massless? By considering one pair of events at a time, find an upper limit on the neutrino mass by asking how large the neutrino mass would need to be to produce an unacceptably large time delay between the detection of neutrinos of different energies. You can neglect the existence of more than one neutrino flavor.

A supernova happens when a massive star has burned all of its nuclear fuel and can no longer withstand the gravitational pressure. When the star is imploding, the electrons and protons in its interior are so close together that they annihilate into $p^+ + e^- \to n + \nu_e$, emitting a huge number of neutrinos. This and other processes produce a huge number of neutrinos in just a few seconds, stealing away a huge amount of energy from inside the star.

Regardless of the mechanism behind the supernova explosion and neutrino creation, we assume that all neutrinos are emitted simultaneously and that some fraction of them travel to Earth, where they are detected. If two neutrinos of the same energy are emitted at the same time, we expect their arrival time to be equal (we are neglecting the physical extent of the exploding star and of the detectors – recall that the distance traveled is much larger than any other distance in the problem). Therefore, under these (spoiler alert: strong!) assumptions, we expect the arrival of neutrinos to depend only on their energy (and therefore, velocity).

First, note that the distance traveled is $L = 50$ kpc $\simeq 1.6 \times 10^{21}$ m $\simeq 5.3 \times 10^{12}$ s in natural units. To travel that "distance", a neutrino with speed v_ν and energy E_ν would take

$$\Delta t = \frac{L}{v_\nu}. \tag{1.16}$$

The neutrino velocity is given by $v_\nu = p_\nu / E_\nu$, and by virtue of the relativistic energy relation in Equation (1.5), we can write it as

$$v_\nu = \frac{\sqrt{E_\nu^2 - m_\nu^2}}{E_\nu} = \sqrt{1 - \frac{m_\nu^2}{E_\nu^2}}. \tag{1.17}$$

Now – I don't like squared roots; they are very awkward mathematical objects. Fortunately, this squared root is very close to one, so I will approximate it as one plus a small correction term. Since it appears in the denominator, all we will need is:

$$\frac{1}{\sqrt{1 - \frac{m_\nu^2}{E_\nu^2}}} \simeq \frac{1}{1 - \frac{m_\nu^2}{2E_\nu^2}} \simeq 1 + \frac{m_\nu^2}{2E_\nu^2} + \text{ corrections of order } \frac{m_\nu^4}{E_\nu^4}. \tag{1.18}$$

If you don't believe me, you can compute $(1-x)^{-n}$ and $1 + x/n$ for a couple of values of $x \ll 1$ and n. This is called a Taylor expansion and will be a very useful tool in your future.

Using the expansion above, we find

$$\Delta t = L \times \left(1 + \frac{m_\nu^2}{2E_\nu^2}\right). \tag{1.19}$$

This is the time that it takes a single neutrino to reach us. We are interested in the delay between two different neutrinos of different energies. The difference in propagation time between pairs of neutrinos under our assumptions is simply the difference in their times of detection, which I will call $t_i - t_j$.

For the first and seconds neutrinos observed, we have

$$t_2 - t_1 = \Delta t_2 - \Delta t_1 = L\left(1 + \frac{m_\nu^2}{2E_2^2}\right) - L\left(1 + \frac{m_\nu^2}{2E_1^2}\right) \qquad (1.20)$$
$$= \frac{Lm_\nu^2}{2}\left(\frac{1}{E_2^2} - \frac{1}{E_1^2}\right),$$

which gives an idea of what the general formula is. For this pair (1 and 2), their arrival time differs by 0.3 s. This difference would be caused by a neutrino mass of

$$m_\nu = \sqrt{\frac{2(t_2 - t_1)}{L}\left(\frac{1}{E_2^2} - \frac{1}{E_1^2}\right)^{-1}} = 1 \times 10^{-7}\ \text{MeV} = 0.1\ \text{eV}, \qquad (1.21)$$

where we used seconds for t and L and MeV for the energies. This is a fantastic result and we will see it is more or less what we believe neutrino masses are. However, it is too soon to declare victory! Let's compute what the neutrino mass prediction is given the other pairs of neutrinos.

Pair 1 and 3) Following the procedure above, now with $t_3 - t_1 = 9$ s and $E_{\nu_3} =$ 10 MeV gives

$$m_\nu \simeq 20\ \text{eV}. \qquad (1.22)$$

That is much larger than the one predicted by the neutrino pair 1 and 2. What happened? Let's investigate one more combination,

Pair 2 and 3) Following the procedure above, now with $t_3 - t_2 = 9 - 0.3 = 8.7$ s and $E_{\nu_3} = 10$ MeV and $E_{\nu_2} = 0.3$ MeV. Clearly, there's something off about this: how can the faster neutrino arrive later? Unfortunately, that is what the experiment found.

This is not perhaps that surprising given that neutrinos are emitted in a duration of a few seconds during the supernova explosion. What we are seeing are actually neutrinos emitted at different times. Our assumptions were too strong, but they still tell us that supernova neutrinos are an invaluable tool to study their properties! The distance traveled is so large that their masses can actually cause a delay, unlike in Earth-based experiments.

2 Second session

Question 1) Neutrinos can oscillate from one flavor to another in a process called neutrino oscillations or flavor transformation. The probability that a neutrino transforms from a flavor α into a flavor β is denoted as $P(\nu_\alpha \to \nu_\beta)$. For instance, $P(\nu_\mu \to \nu_e)$ denotes the probability that a muon neutrino will transform into an electron neutrino after some time. This type of experiment is called an appearance experiment.

In contrast, we denote the probability of "survival" as $P(\nu_\alpha \to \nu_\alpha)$, representing the probability that a neutrino of flavor α will continue to be a flavor α after some time. If this is smaller than one, ν_α neutrinos are said to have disappeared. This type of experiment is called a disappearance experiment.

a) The quantity $P(\nu_\mu \to \nu_\tau)$ can be as large as 1 in certain cases. What does this mean for the quantum numbers L_μ and L_τ in the Standard Model? How about $L_\mu + L_\tau$?

> As we discussed, the fact that muon neutrinos can transform into tau neutrinos means that muon number (L_μ) and tau number (L_τ) are not conserved. In this case, the sum of the two is conserved, however. This is called *lepton universality violation* – meaning neutrinos told us that not all leptons are "born the same"; something in nature seems to prefer some over others.
> This is a small effect, as we will see. So far, it is only visible in neutrino oscillations, but we have been looking for other processes that have this behavior. One recent development was that some collider experiments reported that they saw some violation of universality in a completely different way. They think that the decays of B mesons into electrons and muons are different when these should actually be the same. The exact measurement is the decay rate of $B \to K e^+ e^-$ and $B \to K \mu^+ \mu^-$. It is too early to draw any conclusions, but it is certainly something to keep an eye on.

b) So far we have not measured any amount of $P(\nu_\alpha \to \overline{\nu_\alpha})$ for any $\alpha = e, \mu, \tau$ in experiments. If this probability were non-zero, what would that mean for the lepton sector and for the quantum number L?

> If we observed neutrino oscillations into antineutrinos, we would have observed lepton number violation. We would know that lepton number L is not conserved. So far, this has never been observed in nature, but there are good reasons to believe it exists. The best experimental test of lepton number violation we have so far is called *neutrino-less double beta decay*, where two neutrons undergo radioactive decay simultaneously, creating a pair of electrons, but no neutrinos: $nn \to p^+ p^+ e^- e^-$. This can only happen if lepton number is violated.

c) Under CP conjugation (charge-parity transformation, transforming particles into antiparticles) the probability $P(\nu_\alpha \to \nu_\beta)$ becomes $P(\overline{\nu_\alpha} \to \overline{\nu_\beta})$ and under T reversal (time reversal), the probability $P(\nu_\alpha \to \nu_\beta)$ becomes $P(\nu_\beta \to \nu_\alpha)$. Are disappearance experiments good tests of CP symmetry? (Hint: think about what CPT conservation means for these probabilities.)

Disappearance experiments are not so useful for testing CP conservation because the CP transformation gives the same answer as the CPT transformation. To see that, note that:

$$\begin{aligned} P(\nu_e \to \nu_e) &\xrightarrow{CP} P(\overline{\nu}_e \to \overline{\nu}_e), \\ P(\nu_e \to \nu_e) &\xrightarrow{T} P(\nu_e \to \nu_e), \\ P(\nu_e \to \nu_e) &\xrightarrow{CPT} P(\overline{\nu}_e \to \overline{\nu}_e). \end{aligned} \tag{2.1}$$

Since the transformation is the same as the first, CP is guaranteed not to change the probability in this case.
For appearance, on the other hand,

$$\begin{aligned} P(\nu_e \to \nu_\mu) &\xrightarrow{CP} P(\overline{\nu}_e \to \overline{\nu}_\mu), \\ P(\nu_e \to \nu_\mu) &\xrightarrow{T} P(\nu_\mu \to \nu_e), \\ P(\nu_e \to \nu_\mu) &\xrightarrow{CPT} P(\overline{\nu}_\mu \to \overline{\nu}_e). \end{aligned} \tag{2.2}$$

In this case, CPT does something completely different from CP, and we can hope to observe CP violation in the appearance experiment. We can also look for T violation by comparing $P(\nu_e \to \nu_\mu)$ and $P(\nu_\mu \to \nu_e)$.

Ps. Why is CPT conserved? This is one of the deepest symmetries in particle physics and is intrinsically related to the structure of space-time. If physics does not depend on the position and time coordinates that you find yourself in (something called Lorentz symmetry), then CPT should be conserved. People still look for violations of CPT, but so far, nothing. And that's, in a sense, a good thing. All of our best theories obey CPT and Lorentz symmetry – something outside this paradigm would be incredibly weird.
There is a nice video by Veritasium on this: https://www.youtube.com/watch?v=yArprk0q9eE.

Bonus question 2): If neutrinos have a mass of $m_\nu = 0.1$ eV, for what distance do I need to let them travel before I can measure a delay of 1 ns with respect to photons? This timing resolution is as good as it gets for laboratory neutrino experiments. How does this compare with the Earth's radius? (Hint: If you understand the previous bonus question, this will be just a direct application of Equation (1.19).)

Imagine a beam of photons that takes $t_\gamma = L/c = L$ to go from the source and the detector and a beam of neutrinos that takes a slightly longer time t_ν to travel the same distance L. We can write the delay between the neutrino and the photon as $\Delta t = t_\nu - t_\gamma$, and using Equation (1.19) and natural units,

$$\Delta t = t_\nu - t_\gamma = L\left(1 + \frac{m_\nu^2}{2E_\nu^2}\right) - L \tag{2.3}$$

which means that

$$L = \frac{2E_\nu^2 \Delta t}{m_\nu^2}. \tag{2.4}$$

So the distance we will need depends on the energy of the neutrino. We can take many values, and that won't change the main conclusion: L would need to be very large! For instance, for $E_\nu = 1$ MeV.

$$L = \frac{2 \times 10^{-9}\ \text{s} \times 3 \times 10^{8}\ \text{m/s}}{0.1\ \text{eV}} = 6 \times 10^{15}\ \text{m} \simeq 10^{9} R_\oplus. \tag{2.5}$$

where $R_\oplus \simeq 6,300$ km is the radius of the Earth. Oliver also pointed out that his would be 40,000 times the size of one astronomical units (the distance between the Sun and the Earth). What's more: this would be around 1,000 times larger than the orbit of Pluto!

3 Third session

Preliminaries) Let me start with some definitions. This is a neutrino flavor state:

$$|\nu_\alpha\rangle \,. \tag{3.1}$$

Here the flavor $\alpha = e, \mu$ or τ is determined by the charged lepton with which the neutrino interacted. These are the states that we produce or measure in our experiments because they are the ones that feel the Weak force. However, they do not have a determined mass[1]. When neutrinos propagate, they travel in a superposition of different states called neutrino mass states. They are represented as

$$|\nu_i\rangle \,, \tag{3.2}$$

where $i = 1, 2, 3$ identifies the mass of the neutrino. That is, $|\nu_1\rangle$ has a mass m_1, and $|\nu_2\rangle$ has a mass m_2, and so on. These are the states that experience time and space in the usual way (similarly to an electron or proton).

Let's simplify the problem and assume that there are only two neutrinos. I can count them as flavor states or as mass states: ν_e and ν_μ, or ν_1 and ν_2. When I produce a flavor state in my experiment, I produce a superposition of states:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \,, \tag{3.3}$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \,, \tag{3.4}$$

where the θ is some angle that controls how misaligned the neutrino mass and flavor states are. In this convention, $\theta = 0$ means that the ν_e is just the first mass state.

Recall that these states satisfy the following properties [2]:

$$\langle\nu_e|\nu_e\rangle = 1, \quad \langle\nu_\mu|\nu_\mu\rangle = 1, \text{ and } \langle\nu_e|\nu_\mu\rangle = 0, \tag{3.5}$$

$$\langle\nu_1|\nu_1\rangle = 1, \quad \langle\nu_2|\nu_2\rangle = 1, \text{ and } \langle\nu_1|\nu_2\rangle = 0. \tag{3.6}$$

After some time t has passed, a neutrino mass state *evolves* according to its "internal clock", and changes. Remember, this "internal clock" ticks at different speeds depending on the mass of the neutrino. We denote the time-evolved state as $|\nu_i(t)\rangle$. The mathematical relation is given by

$$|\nu_1(t)\rangle = e^{-iE_{\nu_1}t}\,|\nu_1\rangle \,, \tag{3.7}$$

[1]How is this possible?! Particles are supposed to have one specific mass, right? Yes, that is true. The correct way to think about it is to think of flavor states as made-up mathematical objects and consider only the mass states as physical particles. In this picture, when pions decay for example, they are actually producing all neutrino mass states ($\pi^+ \to e^+\nu_1$, $\pi^+ \to e^+\nu_2$, and $\pi^+ \to e^+\nu_3$), but some mass states are produced more often than others, depending on their affinity with the charged-lepton companion. For instance, the first neutrino, ν_1, has a higher affinity with electrons than ν_2, and so the object $|\nu_e\rangle$ contains more $|\nu_1\rangle$ than $|\nu_2\rangle$. Conversely, we also say that ν_1 has a larger ν_e component than ν_2. Because neutrino masses are so small and close together, the different neutrino mass propagate almost in synchrony and cannot be distinguished given the quantum mechanical uncertainty on their energies and masses. Just like Schröedinger's cat, flavor states have masses m_1, m_2, and m_3 all at once!

[2]can you show it?

and similarly for ν_2. Since $e^{-iE_{\nu_1}t}$ is just a number, we note that

$$\langle \nu_i | \nu_j(t) \rangle = \langle \nu_i | \, e^{-iE_{\nu_j}t} \, | \nu_j \rangle = e^{-iE_{\nu_j}t} \langle \nu_i | \nu_j \rangle = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ otherwise} \end{cases} \tag{3.8}$$

but for time-evolved flavor states, $\langle \nu_\alpha | \nu_\beta(t) \rangle$

Using the properties above, we will calculate the oscillation probability for a muon-neutrino beam to appear as an electron-neutrino beam after some time $t = L/c = L$.

Let's begin by asking how much overlap is there between a ν_e state and the ν_μ state after the latter has evolved for some time t? This is simply given by what we call the "amplitude",

$$A_{\nu_\mu \to \nu_e} = \langle \nu_e | \nu_\mu(t) \rangle = \left(\langle \nu_1 | \cos\theta + \langle \nu_2 | \sin\theta \right) \times \left(-\sin\theta \, | \nu_1(t) \rangle + \cos\theta \, | \nu_2(t) \rangle \right). \tag{3.9}$$

Clearly, for $t = 0$, this should be zero. But for $t \neq 0$, we will need to evolve the mass states. Keeping only the non-zero bra-kets, we get

$$\begin{aligned} A_{\nu_\mu \to \nu_e} &= -\cos\theta \sin\theta \, \langle \nu_1 | \nu_1(t) \rangle + \cos\theta \sin\theta \, \langle \nu_2 | \nu_2(t) \rangle && (3.10) \\ &= -\cos\theta \sin\theta e^{-iE_{\nu_1}t} \langle \nu_1 | \nu_1 \rangle + \cos\theta \sin\theta e^{-iE_{\nu_2}t} \langle \nu_2 | \nu_2 \rangle && (3.11) \\ &= \cos\theta \sin\theta (e^{-iE_{\nu_2}t} - e^{-iE_{\nu_1}t}), && (3.12) \end{aligned}$$

we will use this formula to obtain the probability below.

Question 1) Assume that all mass state neutrinos have roughly the same momentum $p_1 \simeq p_2 \equiv p$, expand the exponent in the complex exponential above using:

$$E = \sqrt{p^2 + m^2} \simeq p \left(1 + \frac{m^2}{2p^2} \right). \tag{3.13}$$

Can you factor out part of the exponential from the sum? You should find something that looks like $\cos\theta \sin\theta e^{(\cdots)}(1 - e^{(\cdots)})$. (Hint: recall that $e^{i(x+y)} = e^{ix}e^{iy}$).

By using the squared root expansion and the properties of the exponential function, we can show

$$\begin{aligned} A_{\nu_\mu \to \nu_e} &= \sin\theta \cos\theta e^{-iE_{\nu_2}t}(1 - e^{i(E_{\nu_2} - E_{\nu_1})t}) && (3.14) \\ &= \frac{\sin 2\theta}{2} e^{-iE_{\nu_2}t}(1 - e^{i\frac{\Delta m^2_{21} t}{2p}}), \end{aligned}$$

where $\Delta m^2_{21} = m^2_2 - m^2_1$. Let me now define the variable for the next part,

$$\Delta_{21} = \frac{\Delta m^2_{21} t}{2p}. \tag{3.15}$$

Question 2) Now, square the amplitude to obtain the probability of oscillations:

$$P(\nu_\mu \to \nu_e) = |A_{\nu_\mu \to \nu_e}|^2. \tag{3.16}$$

Using Euler's identity $e^{ix} = \cos x + i \sin x$, express your answer using only sines and cosines. (Hint: recall $|a|^2 = a^* a$ and that $(e^{ix})^* = e^{-ix}$. You will also need some trigonometry.).

> By using the squared root expansion and the properties of the exponential function ($|e^{ix}| = 1$), we can show
>
> $$\begin{aligned} |A_{\nu_\mu \to \nu_e}|^2 &= \frac{\sin^2 2\theta}{4} \left| e^{-iE_{\nu_2} t}(1 - e^{i\frac{\Delta m^2_{21} t}{2p}}) \right|^2 \\ &= \frac{\sin^2 2\theta}{4} \left| e^{-iE_{\nu_2} t} \right|^2 \left| (1 - e^{i\Delta_{21}}) \right|^2 \\ &= \frac{\sin^2 2\theta}{4} \left| (1 - \cos \Delta_{21} - i \sin \Delta_{21}) \right|^2 \\ &= \frac{\sin^2 2\theta}{4} \left((1 - \cos \Delta_{21})^2 + (\sin \Delta_{21})^2 \right) \\ &= \frac{\sin^2 2\theta}{4} \left(1 + 2 \cos \Delta_{21} + \cos^2 \Delta_{21} + \sin^2 \Delta_{21} \right) \\ &= \sin^2 2\theta \left(\frac{2 + 2\cos \Delta_{21}}{4} \right) \\ &= \sin^2 2\theta \sin^2 \frac{\Delta_{21}}{2} = \sin^2 2\theta \sin^2 \frac{\Delta m^2_{21} t}{2p}, \end{aligned} \tag{3.17}$$
>
> which matches the expression below when $p \to E_\nu$ and $t \to L$.

Question 3) Usually we measure the distance traveled by neutrinos $L = t \times c$ instead of the time, and use energy and momentum interchangeably ($p \sim E_\nu$). You should then find the following form for the probability,

$$P(\nu_\mu \to \nu_e) = \sin^2 2\theta \sin^2 \frac{\Delta m^2_{21} L}{4E_\nu}. \tag{3.18}$$

a) Is this the same as your expression? Rewrite it so that you can use the following units: km for L, GeV for E_ν, and eV^2 for $\Delta m^2_{21} = m^2_2 - m^2_1$. Calculate it for $L = 1000$ km, $\delta m^2_{21} \sim 2 \times 10^{-3}$, and $E_\nu = 5$ GeV assuming $\theta = \pi/4$.

Firstly, it should be clear that in natural units, the argument of the second sine has to be dimensionless (clearly this has to be the case, as otherwise, changing units would change the sine and therefore the probability. Physics is independent of your chosen units!). That means that if Δm^2 is expressed in eV^2, as usual, then L would have to be expressed in 1/eV and p in eV.
We also know that $\hbar c = 1 = 197$ fm.MeV, so $\mathrm{eV}^{-1} = 197 \times 10^{-9}$ m $= 1.97 \times 10^{-10}$ km, so multiplying and dividing by eV^2,

$$\frac{\Delta m_{21}^2 L}{4E_\nu} = \left(\frac{\frac{\Delta m_{21}^2}{\mathrm{eV}^2}\frac{L}{\mathrm{eV}^{-1}}}{4\frac{E_\nu}{\mathrm{eV}}}\right) = 0.25\times\left(\frac{\frac{\Delta m_{21}^2}{\mathrm{eV}^2}\frac{L}{1.97\times 10^{-10}\ \mathrm{km}}}{\frac{E_\nu}{10^{-9}\ \mathrm{GeV}}}\right) = \frac{0.25}{0.197}\times\left(\frac{\frac{\Delta m_{21}^2}{\mathrm{eV}^2}\frac{L}{\mathrm{km}}}{\frac{E_\nu}{\mathrm{GeV}}}\right), \tag{3.19}$$

so we can simply use

$$P(\nu_\mu \to \nu_e) = \sin^2 2\theta \sin^2\left(1.27\frac{\Delta m_{21}^2[\mathrm{eV}^2]L[\ \mathrm{km}]}{E_\nu[\mathrm{GeV}]}\right). \tag{3.20}$$

b) If the mixing angle is large, $\theta = \pi/4$, how large is the maximum and minimum value this probability can take, depending on L/E and Δm^2?

Easy: 1 and 0. For arbitrary values of θ, the minimum is still zero, but the maximum appearance probability is simply $\sin^2 2\theta$.

c) What happens when $L/E \to \infty$? What would the experiment see if its measurement of the neutrino energy is not always accurate?

The neutrinos oscillate very fast and the detector cannot distinguish peaks from the oscillation's throats, so it sees just an averaged-out sinusoidal. Very small changes in E_ν cause a big change in P. From the average of the square of a sine function, we get

$$P(\nu_\mu \to \nu_e) = \frac{1}{2}\sin^2 2\theta. \tag{3.21}$$