1 Assignment 4b — Basics of Early Universe Cosmology

In this assignment, we will derive the some consequences of a FLRW Universe. Let us start from the FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.1}$$

where $a(t)$ is the scale factor and k is the spatial curvature constant. Due to the expansion of the Universe, distances between comoving observers change with time as $r(t) = a(t)r$, where r called the comoving distance (constant in time).

From General Relativity, Einstein's equations relate the geometry of spacetime to its energy content via

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, G is Newton's gravitational constant, $T_{\mu\nu}$ is the energy-momentum tensor, and Λ is the cosmological constant. Assuming a perfect fluid energy-momentum tensor,

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{1.3}$$

where ρ is the energy density, p is the pressure, and u_μ is the four-velocity of the fluid. In particular, in the rest frame of the background fluid, $u^\mu = (1, 0, 0, 0)$ and the energy-momentum tensor reduces to $T_{\mu\nu} = \mathrm{diag}(\rho, p, p, p)$. Using the FLRW metric, the perfect fluid form of $T_{\mu\nu}$, and all the Christoffel symbols, Einstein's equations can be expanded to obtain the Friedmann equations.

The *Friedmann equations* for a homogeneous and isotropic Universe are

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \end{aligned} \tag{1.4}$$

where we defined the Hubble parameter $H = \frac{\dot{a}}{a}$, ρ is the energy density, p is the pressure, and Λ is the cosmological constant. It is also useful to write the second Friedmann equation in terms of $\dot{H} = \frac{\ddot{a}}{a} - H^2$ as

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2}. \tag{1.5}$$

We will not derive these equations here, but you can find the derivation in many standard textbooks on cosmology, e.g., *Modern Cosmology* by Scott Dodelson.

(**1 point**) 1. By inspecting the FLRW metric and the definition of the Hubble parameter, prove Hubble's law, which states that the recessional velocity v of a distant galaxy is proportional to its distance d from us, i.e., $v = \dot{r}(t) = Hr(t)$, where H is the Hubble constant (technically, only a constant at present times, as you will show below).

(**3 points**) 2. Derive the continuity equation for the energy density ρ and pressure p from the Friedmann equations

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{1.6}$$

(**3 points**) 3. Consider a Universe dominated by a single fluid that respects the equation of state $w = p/\rho$ (i.e., $p = w\rho$). By integrating the continuity equation, find how the energy density ρ scales with the scale factor $a(t)$. Take w to be constant.

(**3 points**) 4. Using the result from Question 3, solve the first Friedmann equation for a flat Universe ($k = 0$) to find how the scale factor $a(t)$ evolves with time for the following cases:

   (a) A radiation-dominated Universe ($w = 1/3$, $\Lambda = 0$).

   (b) A matter-dominated Universe ($w = 0$, $\Lambda = 0$).

   (c) A Universe dominated by a negative pressure fluid with $w = -1$ (you can either consider a ρ density with $w = -1$ and set $\Lambda = 0$ or consider a universe with just a cosmological constant, $\Lambda > 0$, and nothing else, $\rho = 0$. The qualitative result should be the same).

   You are calculating how the expansion of the Universe changes with time in each case. If we know what is inside the Universe, we can predict how fast it expands!

(**Bonus 4 points**) 5. Consider a flat Universe ($k = 0$) that has both matter ($w = 0$) and radiation ($w = 1/3$) components, with energy densities ρ_m and ρ_r, respectively. Using the results from Question 3, write down the first Friedmann equation for this Universe. Show that there is a time $t_{\rm eq}$ when the energy densities of matter and radiation are equal, i.e., $\rho_m(t_{\rm eq}) = \rho_r(t_{\rm eq})$. Find an expression for $t_{\rm eq}$ in terms of the present-day energy densities $\rho_{m,0}$ and $\rho_{r,0}$ (i.e., the energy densities at $a = 1$). Knowing that $\rho_{m,0} \approx 0.3\rho_c$ and $\rho_{r,0} \approx 5 \times 10^{-5}\rho_c$, where $\rho_c = 3H_0^2/(8\pi G)$ is the critical density[1] and $H_0 = 70\,\mathrm{km/s/Mpc}$ is the Hubble constant today, estimate the time of matter-radiation equality $t_{\rm eq}$ in years and in redshift $z_{\rm eq} = \frac{a(t_0)}{a(t_{\rm eq})} - 1$.

[1] The critical density is the energy density required for a flat Universe ($k = 0$).