1 Noether's Theorem in Quantum Mechanics

In the first question of the exam I brought up the concept of symmetries in Quantum Mechanics. The solution to Question 1c is an extremely simple proof of **Noether's theorem**. here, we go through it and explore a few more details about this.

For a real continuous parameter λ, we define the unitary operator

$$\hat{W} = e^{-\frac{i}{\hbar}\lambda \hat{G}}. \tag{1}$$

We say that $\hat{W}$ is a (continuous) symmetry transformation if it maps solutions to the Schrödinger equation to other solutions of the Schrödinger equation. That is, $\hat{W}$ is a continuous symmetry if

$$i\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad \iff \quad i\hbar\partial_t(\hat{W} |\psi(t)\rangle) = \hat{H}(\hat{W} |\psi(t)\rangle). \tag{2}$$

To prove the statement in Question 1c, start with the right-hand side equation. Use the product rule and do

$$i\hbar\partial_t(\hat{W} |\psi(t)\rangle) = i\hbar(\partial_t\hat{W}) |\psi(t)\rangle + \hat{W}\left(i\hbar\partial_t |\psi(t)\rangle\right) = i\hbar(\partial_t\hat{W}) |\psi(t)\rangle + \hat{W}\hat{H} |\psi(t)\rangle \tag{3}$$

$$= \hat{H}(\hat{W} |\psi(t)\rangle). \tag{4}$$

Rearranging,

$$i\hbar(\partial_t\hat{W}) |\psi(t)\rangle = [\hat{H}, \hat{W}] |\psi(t)\rangle . \tag{5}$$

Note that this equation holds for any state $|\psi(t)\rangle$, and therefore we get the operator identity[1]

$$i\hbar\partial_t\hat{W} = [\hat{H}, \hat{W}]. \tag{6}$$

Equivalently, we can write the same operator equation in Heisenberg picture by multiplying Eq. (5) by $\hat{U}(t)^\dagger$ from the left:

$$i\hbar\left(\hat{U}(t)^\dagger\partial_t\hat{W}\hat{U}(t)\right) |\psi(0)\rangle = \left(\hat{U}(t)^\dagger[\hat{H}, \hat{W}]\hat{U}(t)\right) |\psi(0)\rangle \tag{7}$$

which then gives

$$\left(\partial_t\hat{W}(t)\right)_H = -\frac{[\hat{W}_H(t), \hat{H}]}{i\hbar}, \tag{8}$$

where I picked up the minus sign by switching the order of the commutator. This form, in particular, is particularly useful.

If we look at the Heisenberg equation of motion, it says that

$$\boxed{i\hbar\frac{d\hat{W}_H(t)}{dt} = i\hbar\left(\frac{\partial\hat{W}(t)}{\partial t}\right)_H + [\hat{W}_H(t), \hat{H}] = 0.} \tag{9}$$

In other words, a symmetry operator $\hat{W}$ that satisfies Eq. (2) corresponds to a conserved quantity in the Heisenberg picture.

We can go further than Eq. (9) and apply the time derivative to the Heisenberg form of the operator $\hat{W}_H(t) = e^{-\frac{i}{\hbar}\lambda\hat{G}_H}$, where $\hat{G}_H$ is the generator of the symmetry in the Heisenberg picture and λ is typically time-independent. Acting the derivative, we get

$$\frac{d}{dt}\hat{W}_H(t) = 0 \quad \implies \quad -\frac{i}{\hbar}\left(\lambda\frac{d\hat{G}_H(t)}{dt}\right)\hat{W}_H(t) = 0 \quad \implies \quad \boxed{\frac{d\hat{G}_H(t)}{dt} = 0.} \tag{10}$$

This is the standard result of **Noether's theorem** in the context of quantum mechanics.

[1] Note that since our states evolve unitarily, the value of the basis kets at any time span the entire Hilbert space at any other time.

The generator of a continuous symmetry is a conserved quantity, and every conserved quantity generates a continuous symmetry.

Technically, the derivation above assumes $[\hat{G}, d\hat{G}/dt] = 0$, but the result is valid even when this commutator is nonzero. To see that, use $\partial_\lambda \hat{W}_H(t) = -\frac{i}{\hbar}\hat{G}_H(t)\hat{W}_H(t)$ and note that if $\hat{W}_H(t)$ is conserved, then $\partial_\lambda \partial_t \hat{W}_H(t) = 0$, which then swapping the order of derivatives gives $\partial_t \partial_\lambda \hat{W}_H(t) = \partial_t \left(-\frac{i}{\hbar}\hat{G}_H(t)\hat{W}_H(t)\right) = 0$, implying that $\frac{d\hat{G}_H(t)}{dt} = 0$ in all generality.

2 Examples

2.1 Translations

If the system has a Hamiltonian that is invariant under spatial translations in a given direction $\mathbf{n}$, the generator of the symmetry, the component of the momentum along that direction, $\hat{\mathbf{p}} \cdot \mathbf{n}$, is conserved in the Heisenberg picture:

$$\mathcal{J}_{\hat{n}}(\Delta x) = e^{-\frac{i}{\hbar}\Delta x \hat{\mathbf{p}}\cdot\mathbf{n}} \implies \frac{d(\hat{\mathbf{p}} \cdot \mathbf{n})_H(t)}{dt} = 0. \tag{11}$$

This statement we already derived in the Ehrenfest theorem. Recall that for a particle in a potential $V(\mathbf{x})$,

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}) \tag{12}$$

With the Heisenberg equation of motion, we have

$$\frac{d}{dt}\langle\hat{\mathbf{p}}\rangle = \langle\frac{[\hat{\mathbf{p}}, \hat{H}]}{i\hbar}\rangle = \langle\frac{[\hat{\mathbf{p}}, V(\hat{\mathbf{x}})]}{i\hbar}\rangle = -\langle\nabla V(\hat{\mathbf{x}}(t))\rangle \tag{13}$$

where we used the same relation $[\hat{\mathbf{p}}, F(\hat{\mathbf{x}})] = -i\hbar\nabla F(\hat{\mathbf{x}})$ (proved below). In other words, momentum is conserved if the potential is translationally invariant, i.e., if it does not depend on the position along the direction of interest. Being a little more careful, we would find $\hat{\mathbf{p}} \cdot \mathbf{n}$ is conserved if and only if $\mathbf{n} \cdot \nabla V = 0$.

For a differentiable function F, we can evaluate the commutator by acting on an arbitrary position-space wavefunction $\psi(\mathbf{x})$. Using $\langle\mathbf{x}|\,\hat{\mathbf{p}}\,|\psi\rangle = -i\hbar\nabla\psi(\mathbf{x})$ and the product rule,

$$\begin{aligned}\langle\mathbf{x}|\,[\hat{\mathbf{p}}, F(\hat{\mathbf{x}})]\,|\psi\rangle &= -i\hbar\nabla(F(\mathbf{x})\psi(\mathbf{x})) + i\hbar F(\mathbf{x})\nabla\psi(\mathbf{x}) \\ &= -i\hbar\left(\nabla F(\mathbf{x})\right)\psi(\mathbf{x}).\end{aligned} \tag{14}$$

Since this holds for arbitrary ψ in the common domain of the operators, we obtain the operator identity

$$\boxed{[\hat{\mathbf{p}}, F(\hat{\mathbf{x}})] = -i\hbar\left(\nabla F\right)(\hat{\mathbf{x}}).} \tag{15}$$

2.2 Rotations

The generator of rotations in 3D is the angular momentum operator $\hat{\mathbf{J}}$. If the system is invariant under rotations around an axis $\mathbf{n}$, the component of the angular momentum along that axis is conserved in the Heisenberg picture:

$$\mathcal{R}_{\mathbf{n}}(\theta) = e^{-\frac{i}{\hbar}\theta\hat{\mathbf{J}}\cdot\mathbf{n}} \implies \frac{d(\hat{\mathbf{J}} \cdot \mathbf{n})_H(t)}{dt} = 0. \tag{16}$$

This again is obviously the case for a free particle. But it is not true for our spin-1/2 particle in a B field in the z direction since that broke rotational symmetry around the x and y axes. Note that in that case, with $\hat{H} = \omega \hat{S}_z$, the z component of angular momentum (just spin in our spin-1/2 example) is conserved,

$$\frac{d\hat{S}_{z,H}(t)}{dt} = 0, \tag{17}$$

but as you found in Question 5 of HW6, the x and y components of spin are not conserved:

$$\frac{d\hat{S}_{x,H}(t)}{dt} = -\omega \hat{S}_{y,H}(t), \qquad \frac{d\hat{S}_{y,H}(t)}{dt} = \omega \hat{S}_{x,H}(t).$$

2.3 Time Translations

The time-*translation* (not evolution) operator $\hat{T}(\tau) = e^{-\frac{i}{\hbar}\tau \hat{H}}$ acts as

$$\hat{T}(\tau)\,|\psi(t)\rangle = e^{-\frac{i}{\hbar}\tau \hat{H}}\,|\psi(t)\rangle = e^{-\frac{i}{\hbar}(t+\tau)\hat{H}}\,|\psi(0)\rangle = |\psi(t+\tau)\rangle \tag{18}$$

for any fixed time interval τ. Note that the parameter $\lambda = \tau$ is a fixed time interval, so $d\tau/dt = 0$ (our assumption that λ was a real constant is still valid). With time-translation symmetry, the total energy of the system is conserved:

$$\hat{T}(\tau) = e^{-\frac{i}{\hbar}\tau \hat{H}} \implies \frac{d\hat{H}_H(t)}{dt} = 0. \tag{19}$$

I am ignoring the more complex time-ordered, integral form of time-evolution here for simplicity.

2.4 Galilean Boosts

Using the boost operator of Question 3, one can show that if the system is invariant under boosts along a certain direction, the corresponding component of the momentum is conserved in the Heisenberg picture:

$$\hat{B}_{\mathbf{n}}(v) = e^{-\frac{i}{\hbar} v \mathbf{K}\cdot\hat{\mathbf{n}}} \quad \implies \quad \frac{d\hat{\mathbf{K}}_H(t)}{dt} = 0. \tag{20}$$

This one is a bit less intuitive, but to see what a consequence of the conservation of $\hat{K} = m\hat{x} - t\hat{p}$, we can write it as (dropping the hats from the operators and the H subscript)

$$\dot{K} = m\dot{x} - t\dot{p} - p = 0 \tag{21}$$

and using the single particle in 1D, we have $\dot{x} = p/m$ and $\dot{p} = -\partial_x V(x)$, so

$$\dot{K} = t\partial_x V(x) = 0 \implies \partial_x V(x) = 0, \tag{22}$$

which is actually the same condition we found for conservation of momentum and translational symmetry.