Quantum Mechanics I HW 6

Pick 4 out of the 5 questions to answer

Question 1)

a) Starting from the definition of the time-dependent operator in the Heisenberg picture, derive the Heisenberg equation of motion. You can assume $\hat{H}$ is time-independent.

b) By taking the expectation value of the Heisenberg equation of motion, write the corresponding equation for the expectation value of the operator. Does it depend on what picture (Heisenberg or Schrödinger) you are using?

c) Let $\hat{W}(\lambda) = e^{-\frac{i}{\hbar}\lambda\hat{G}}$, with $\hat{G}$ Hermitian and possibly explicitly time-dependent. We call $\hat{W}$ a symmetry if, for every λ, it maps solutions of the Schrödinger equation to the same or other solutions of the Schrödinger equation. That is, if $|\psi(t)\rangle$ is a solution, then $\hat{W}(\lambda)\,|\psi(t)\rangle$ should also be a solution for $\hat{W}(\lambda)$ to be called a symmetry. **Without using the Heisenberg picture**, show that $\hat{W}$ is a symmetry if $i\hbar\,\partial_t\hat{W} = [\hat{H}, \hat{W}]$. Recall that a solution respects:

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle\,.$$

Is this consistent with the Heisenberg equation of motion?

Question 2) Neutrinos oscillate between three flavors ν_e, ν_μ, and ν_τ. The flavor states are given by

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U^*_{\alpha i}\,|\nu_i\rangle\,, \quad \alpha = e, \mu, \tau, \quad i = 1, 2, 3 \tag{1}$$

where U is the unitary mixing matrix that relates the flavor eigenstates $|\nu_\alpha\rangle$ to the mass/energy eigenstates $|\nu_i\rangle$. In the energy basis, $\hat{H}\,|\nu_i\rangle = E_i\,|\nu_i\rangle$. Assume a non-degenerate spectrum of energies, $E_i \neq E_j$ for $i \neq j$.

a) Write the probability for a neutrino to oscillate from an initial flavor state $|\nu_\alpha\rangle$ to a final flavor state $|\nu_\beta\rangle$ at a later time t. Express your answer only in terms of the mixing matrix elements $U_{\alpha i}$ and complex exponentials of differences of energies, $\Delta E_{ij} = E_i - E_j$.

b) What is the oscillation probability when the matrix $U_{\alpha i}$ is diagonal?

c) The IceCube experiment detects neutrinos from astrophysical sources, megaparsecs to maybe gigaparsecs away. Using your answer from part (a), do you expect IceCube to observe neutrinos in the same flavor composition as they were produced?

d) Assume that the unitary mixing matrix is fully democratic

$$|U_{\alpha i}| = \frac{1}{\sqrt{3}} \quad \text{for all } \alpha \text{ and } i. \tag{2}$$

Averaging over time at $t \to \infty$, what is the expected flavor composition of neutrinos detected by IceCube as a ratio of $\nu_e : \nu_\mu : \nu_\tau$? Does it depend on the initial flavor composition?

SEE NEXT PAGE

Question 3) Consider a free particle in a one-dimensional space:

$$\hat{H} = \frac{\hat{p}^2}{2m}. \tag{3}$$

The generator of Galilean boosts is given by

$$\hat{K} = m\hat{x} - t\hat{p}, \tag{4}$$

with t time, $\hat{x}$ the position operator, and $\hat{p}$ the momentum operator.

a) Start by considering $\hat{K}$ as a generator of a unitary transformation (Galilean boost). Show that for small boosts by a small velocity v, the momentum operator transforms as

$$\left(\mathbb{1} - \frac{i}{\hbar}v\hat{K}\right)\hat{p}\left(\mathbb{1} + \frac{i}{\hbar}v\hat{K}\right) = \hat{p} + vm + \mathcal{O}(v^2) \tag{5}$$

and the position operator transforms as

$$\left(\mathbb{1} - \frac{i}{\hbar}v\hat{K}\right)\hat{x}\left(\mathbb{1} + \frac{i}{\hbar}v\hat{K}\right) = \hat{x} + vt + \mathcal{O}(v^2) \tag{6}$$

b) Consider $\hat{x}$, $\hat{p}$, $\hat{H}$, and $\hat{K}$. Which of these Heisenberg picture operators are constant in time for the free particle?

c) Using your result above and the initial conditions for $\hat{x}(0) = \hat{x}_0$ and $\hat{p}(0) = \hat{p}_0$, where

$$\hat{K}_0 = m\hat{x}_0 \tag{7}$$

determine $\hat{x}(t)$ in terms of $\hat{p}_0$, $\hat{x}_0$, m, and t.

Question 4)

a) **Briefly** describe the Stern-Gerlach experiment and how it selects spin states.

b) Starting from an unpolarized collection of spins traveling in the y direction, we place a Stern-Gerlach filter along the z-axis (SG$z(+)$). What spin eigenstate is selected by this filter?

c) We now place a Stern-Gerlach filter along the x-axis (SG$x(+)$). How does the intensity of the beam change compared to the beam that **enters** the SG$z(+)$ filter?

d) Now place a third Stern-Gerlach filter along the z-axis:

$$\text{SG}z(+) \to \text{SG}x(+) \to \text{SG}z(-) \tag{8}$$

What is the beam intensity after passing through the third filter compared to the initial unpolarized beam? What happens if we remove the second filter (SG$x(+)$)?

Question 5) Consider the two-level system of a spin-1/2 particle in a magnetic field pointing along the z-axis:

$$\hat{H} = \omega\hat{S}_z = \frac{\omega}{2}\hbar\sigma_z, \quad \omega = |e|B/m_e,$$

where B is the magnitude of the magnetic field.

a) For the above Hamiltonian, working in the Heisenberg picture, write the equation of motion for the spin operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$. Recall the Heisenberg equation of motion for an operator $\hat{O}$:

$$\frac{d\hat{O}}{dt} = \frac{1}{i\hbar}[\hat{O}, \hat{H}] + \frac{\partial \hat{O}}{\partial t}.$$

b) Differentiate the spin operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ again and solve the second-order differential equations to find $\hat{S}_x(t)$, $\hat{S}_y(t)$, and $\hat{S}_z(t)$ as a function of $\hat{S}_x(0)$, $\hat{S}_y(0)$, and $\hat{S}_z(0)$. You will need the first and second derivatives of the spin operators to simplify things.

c) Discuss your result: what is the Bloch vector of this system and how does it evolve in time for this spin-1/2 particle in the magnetic field?

Formulae

- Any two-state system state can be as

$$|\psi\rangle = \cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle$$

where $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$.

- $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$, with ϵ_{ijk} the Levi-Civita symbol ($i, j, k = x, y, z$, $\epsilon_{xyz} = +1$).
- $S_i = \frac{\hbar}{2}\sigma_i$, with σ_i the Pauli matrices. $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
- Time evolution operator:

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}.$$

- Heisenberg picture time evolution:

$\hat{O}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{O}_S e^{-\frac{i}{\hbar}\hat{H}t}$ where $\hat{O}_S$ is the Schrödinger picture operator.

- Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle, \quad i\hbar\frac{\partial}{\partial t}\hat{U}(t) = \hat{H}\hat{U}(t)$$

- Heisenberg equation of motion:

$$\frac{d\hat{O}_H(t)}{dt} = \frac{1}{i\hbar}[\hat{O}_H(t), \hat{H}] + \left(\frac{\partial \hat{O}(t)}{\partial t}\right)_H$$

1 Solutions

Question 1)

a) Applying the time derivative to the Heisenberg picture operator definition:

$$\frac{d}{dt}\hat{O}_H(t) = \frac{d}{dt}\left(\hat{U}(t)^\dagger \hat{O}_S \hat{U}(t)\right) = \hat{U}(t)^\dagger \frac{\partial \hat{O}_S}{\partial t}\hat{U}(t) + \frac{d\hat{U}(t)^\dagger}{dt}\hat{O}_S\hat{U}(t) + \hat{U}(t)^\dagger\hat{O}_S\frac{d\hat{U}(t)}{dt} \tag{9}$$

and using Schrödinger equation for the time evolution operator,

$$\frac{d\hat{U}(t)}{dt} = -\frac{i}{\hbar}\hat{H}\hat{U}(t), \quad \frac{d\hat{U}(t)^\dagger}{dt} = \frac{i}{\hbar}\hat{U}(t)^\dagger\hat{H}. \tag{10}$$

we get

$$\begin{aligned}\frac{d}{dt}\hat{O}_H(t) &= \hat{U}(t)^\dagger\frac{\partial\hat{O}_S}{\partial t}\hat{U}(t) + \frac{i}{\hbar}\hat{U}(t)^\dagger\hat{H}\hat{O}_S\hat{U}(t) - \frac{i}{\hbar}\hat{U}(t)^\dagger\hat{O}_S\hat{H}\hat{U}(t) &(11)\\ &= \hat{U}(t)^\dagger\frac{\partial\hat{O}_S}{\partial t}\hat{U}(t) + \frac{i}{\hbar}\hat{U}(t)^\dagger[\hat{H},\hat{O}_S]\hat{U}(t)\\ &= \frac{1}{i\hbar}[\hat{O}_H(t),\hat{H}] + \left(\frac{\partial\hat{O}(t)}{\partial t}\right)_H &(12)\end{aligned}$$

Note the last term is the Heisenberg picture representation of the explicit time derivative of the operator. The time evolution operator sandwich sits on the outside of the derivative.

---

b) Now the expectation value. The time derivative $\hat{D}_H \equiv d\hat{O}_H(t)/dt$ is just another operator, so it must be true that

$$\langle D_H\rangle_\psi = \langle\psi(0)|\,\hat{D}_H\,|\psi(0)\rangle = \langle\psi(0)|\,\frac{d\hat{O}_H(t)}{dt}\,|\psi(0)\rangle = \frac{d}{dt}\,\langle\psi(0)|\,\hat{O}_H(t)\,|\psi(0)\rangle = \frac{d}{dt}\langle\hat{O}_H(t)\rangle_\psi = \frac{d}{dt}\langle\hat{O}_S\rangle_{\psi(t)}. \tag{13}$$

Note how we passed the kets through the time derivative since they are the frozen Heisenberg picture kets. This alone is enough to show that the Heisenberg equation in terms of expectation values must be the same, but let us look at the right-hand side explicitly:

$$\langle\psi(0)|\left(\frac{1}{i\hbar}[\hat{O}_H(t),\hat{H}] + \left(\frac{\partial\hat{O}(t)}{\partial t}\right)_H\right)|\psi(0)\rangle = \langle\psi(0)|\,\hat{U}(t)^\dagger\left(\frac{1}{i\hbar}[\hat{O}_S,\hat{H}] + \frac{\partial\hat{O}_S}{\partial t}\right)\hat{U}(t)\,|\psi(0)\rangle \tag{14}$$

which again is the same in Heisenberg or Schrödinger picture.

---

c) **See lecture notes on Noether's Theorem.**

Question 2)

a) The time evolution is

$$|\nu_\alpha(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\,|\nu_\alpha(0)\rangle = \sum_i U^*_{\alpha i} e^{-\frac{i}{\hbar}E_i t}\,|\nu_i\rangle\,, \tag{15}$$

so the probability

$$P_{\alpha\to\beta}(t) = |\,\langle\nu_\beta|\nu_\alpha(t)|\nu_\beta|\nu_\alpha(t)\rangle\,|^2 = \left|\sum_i U_{\beta i}U^*_{\alpha i}e^{-\frac{i}{\hbar}E_i t}\right|^2 = \sum_{i,j} U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}e^{-\frac{i}{\hbar}(E_i-E_j)t}. \tag{16}$$

---

b) If the mixing matrix is diagonal, then there must be no oscillations because the flavor and the mass basis coincide (up to an overall phase).[1] Since the mass basis is the energy basis, which is stationary, the flavor states will not evolve non-trivially in time, so no oscillations.

If you want to prove this explicitly starting from part a), we can do

$$\begin{aligned} P_{\alpha\to\beta}(t) &= \sum_{i,j} U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}e^{-\frac{i}{\hbar}(E_i-E_j)t} && (17)\\ &= \sum_{i,j} \delta_{\beta i}\delta^*_{\alpha i}\delta^*_{\beta j}\delta_{\alpha j}e^{-\frac{i}{\hbar}(E_i-E_j)t} && (18)\\ &= \sum_{i} \delta_{\beta i}\delta^*_{\alpha i}\delta^*_{\beta i}\delta_{\alpha i}e^{-\frac{i}{\hbar}(E_i-E_i)t} && (19)\\ &= \sum_{i} \delta_{\beta i}\delta^*_{\alpha i}\delta^*_{\beta i}\delta_{\alpha i} = \delta_{\alpha\beta}. && (20) \end{aligned}$$

---

c) Neutrinos will oscillate and therefore, the flavor composition will change between the source and the observer here at Earth.



Note that at $t\to\infty$, the oscillation probabilities will average out because the average of $e^{-\frac{i}{\hbar}(E_i-E_j)t}\to\delta_{ij}$. One can simplify the oscillation probabilities at long times by averaging over the rapidly oscillating phases:

$$P_{\alpha\to\beta}\to\sum_i U_{\beta i}U^*_{\alpha i}U^*_{\beta i}U_{\alpha i} = \sum_i |U_{\beta i}|^2|U_{\alpha i}|^2. \tag{21}$$

---

d) If the mixing matrix is fully democratic, then all oscillation probabilities are equal, i.e.,

$$P_{\alpha\to\beta}(t) = \sum_i |U_{\beta i}|^2|U_{\alpha i}|^2 = \frac{1}{3}, \tag{22}$$

Now, if we start with a given arbitrary flavor fraction at the source:

$$(f^S_e, f^S_\mu, f^S_\tau), \tag{23}$$

normalized, for instance, such that

$$f^S_e + f^S_\mu + f^S_\tau = 1. \tag{24}$$

[1] Technically, we have that $U_{\alpha i} = e^{\phi_\alpha}\delta_{\alpha i}$ because by unitarity, the diagonal must be filled with numbers such that $|U_{\alpha i}| = 1$ since $\sum_i |U_{\alpha i}|^2 = 1$ as a consequence of unitarity, $U^\dagger U = UU^\dagger = \mathbb{1}$.

Then,

$$f_\beta^\oplus = \sum_\alpha P_{\alpha\to\beta} f_\alpha^S = \sum_\alpha \frac{1}{3} f_\alpha^S = \frac{1}{3}\sum_\alpha f_\alpha^S = \frac{1}{3}. \tag{25}$$

So at the Earth,

$$(f_e^\oplus, f_\mu^\oplus, f_\tau^\oplus) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{26}$$

No matter the flavor composition at the source, we end up with the same flavor composition at the Earth: an equal mixture of all flavors.

Question 3)

a) Working at the order $\mathcal{O}(v)$, we discard any $\mathcal{O}(v^2)$ terms and explicitly calculate the products:

$$\left(\mathbb{1} - \frac{i}{\hbar} v\hat{K}\right)\hat{p}\left(\mathbb{1} + \frac{i}{\hbar} v\hat{K}\right) = \mathbb{1}\hat{p} - \frac{i}{\hbar} v\hat{K}\hat{p} + \frac{i}{\hbar} v\hat{p}\hat{K} + \mathcal{O}(v^2) = \hat{p} - \frac{i}{\hbar} v[\hat{K}, \hat{p}] + \mathcal{O}(v^2). \tag{27}$$

Using $[\hat{x}, \hat{p}] = i\hbar$, a simple calculation gives the desired result by virtue of $[\hat{K}, \hat{p}] = i\hbar m$,

$$\left(\mathbb{1} - \frac{i}{\hbar} v\hat{K}\right)\hat{p}\left(\mathbb{1} + \frac{i}{\hbar} v\hat{K}\right) = \hat{p} + vm + \mathcal{O}(v^2). \tag{28}$$

Same thing for $\hat{x}$, using $[\hat{K}, \hat{x}] = it\hbar$, we get

$$\left(\mathbb{1} - \frac{i}{\hbar} v\hat{K}\right)\hat{x}\left(\mathbb{1} + \frac{i}{\hbar} v\hat{K}\right) = \hat{x} + vt + \mathcal{O}(v^2). \tag{29}$$

---

b) An operator is a constant in time if (dropping the H subscript for the Heisenberg picture):

$$\frac{d\hat{O}}{dt} = 0, \tag{30}$$

where $\hat{O}$ is the operator in the Heisenberg picture. Using Heisenberg's equation of motion, this is equivalent to

$$[\hat{O}, \hat{H}] = -i\hbar\left(\frac{\partial \hat{O}}{\partial t}\right) \tag{31}$$

For the explicitly time-independent operators, we can ask: do they commute with the free-particle hamiltonian $\hat{H} = \hat{p}^2/2m$? Clearly $\hat{p}$, $\hat{p}^2$, and any function of $\hat{p}$ commute with the free-particle hamiltonian $\hat{H} = \hat{p}^2/2m$, while $\hat{x}$ does not because $[\hat{x}, \hat{H}] = [\hat{x}, \hat{p}^2/2m] \neq 0$. For $\hat{K}$, we have to account for the explicit time dependence:

$$\frac{d\hat{K}}{dt} = \frac{[\hat{K}, \hat{H}]}{i\hbar} + \left(\frac{\partial \hat{K}}{\partial t}\right) = \frac{i\hbar\hat{p}}{i\hbar} - \hat{p} = 0. \tag{32}$$

where the commutator

$$[\hat{K}, \hat{H}] = [m\hat{x} - t\hat{p}, \hat{p}^2/2m] = m[\hat{x}, \hat{p}^2/2m] - t[\hat{p}, \hat{p}^2/2m] = m\frac{i\hbar}{2m}2\hat{p} = i\hbar\hat{p}. \tag{33}$$

So, yes, $\hat{K}$ is indeed a constant of motion despite having explicit time dependence.

---

c) Using that $\hat{x}(0) = x_0$ and $\hat{p}(0) = p_0$, and $\hat{K}_0 = m\hat{x_0}$ at $t = 0$, we then have to determine the

time evolution of $x(t)$. There are two ways of doing this. You can go back to our notes on the Ehrenfest theorem, and just follow that derivation which uses two facts:

$$\frac{d\hat{x}}{dt} = \frac{\hat{p}}{m}, \quad \frac{d\hat{p}}{dt} = 0. \tag{34}$$

OR, you can use the fact that $\hat{K}$ is a constant of motion and write

$$\hat{K}(t) = m\hat{x}(t) - t\hat{p}(t) \stackrel{!}{=} \hat{K}_0 = m\hat{x}_0. \tag{35}$$

from which we can solve for $\hat{x}(t)$:

$$\hat{x}(t) = \hat{x}_0 + \frac{\hat{p}_0}{m}t. \tag{36}$$

Question 4)

a) Many ways to describe the experiment, but the most important point here is that a SG experiment is able to split a spin-1/2 system into two beams, one with spin up and one with spin down along the direction of the magnetic field gradient. Up and down refer to the spin projection along the direction of the magnet (typically we place the z axis along the direction of the first magnet).

---

b) We select the $|+\rangle$ state. In our matrix language, recall that

$$|+\rangle \quad \mapsto \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{37}$$

but you don't need to use this form to answer the question.

---

c) Placing a magnet along the x direction will split the beam into two components corresponding to the spin projection along the x axis, i.e., the $|S_x;+\rangle$ and $|S_x;-\rangle$ states, but how do we see that?

Recall that the Stern-Gerlach eigenstates are exactly the form of the most general 2-state system, with the direction of the magnet determining the polar and azimuthal angles of the spinor on the Bloch sphere. In other words, for $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the eigenstates of the spin projection along $\hat{n}$ are

$$|S_{\hat{n}};+\rangle = \cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle, \quad |S_{\hat{n}};-\rangle = \sin\frac{\theta}{2}|+\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\rangle. \tag{38}$$

For the x direction, we have $\theta = \pi/2$ and $\phi = 0$, so the eigenstates along the x axis are

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |S_x;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \tag{39}$$

You didn't really need to know the exact form of the states, but what you should know is this: a state in an orthogonal direction of the a new axis will be an equal superposition of the eigenstates along the original axis, that is,

$$|S_x;+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle), \quad |S_x;-\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle). \tag{40}$$

The signs don't matter much, but the key fact is that the beam that exits the SGx magnet will be split into two equal parts corresponding to the $|S_x;+\rangle$ and $|S_x;-\rangle$ states, and **with respect to the unpolarized beam that enters SGz(+), we have**

$$P = \frac{1}{2} \times |\langle S_x;+|+\rangle|^2 = \frac{1}{2} \times |\frac{1}{\sqrt{2}}|^2 = \frac{1}{4}. \tag{41}$$

with the factor of 1/2 due to the fact that the unpolarized beam entering SG$z(+)$ has half of its intensity blocked. Then the beam that exits the SG$x(+)$ filter is again half as intense, so with respect to the initial unpolarized beam, the intensity is 1/4 of the original.

---

d) Same as above, but now we split $|S_x;+\rangle$ along the z direction again and measure only $|S_z;-\rangle = |-\rangle$. We have the total intensity

$$P = \frac{1}{2} \times |\langle S_x;+|+\rangle|^2 \times |\langle S_z;-|S_x;+\rangle|^2 = \frac{1}{2} \times |\frac{1}{\sqrt{2}}|^2 \times |\frac{1}{\sqrt{2}}|^2 = \frac{1}{8}. \tag{42}$$

If we remove the middle SGx filter, the key observation is that the state that is selected by the filter SG$z(-)$ is **orthogonal** to the state that exits SG$z(+)$. In other words,

$$P = \frac{1}{2} \times |\langle S_z;-|S_z;+\rangle|^2 = 0. \tag{43}$$



This entire question can also be solved by using the analogy with light polarization and taking half of the angle between magnets as the full angle between polarimeters. That is, SG$z(+)$ followed by SG$x(+)$ is equivalent to a polarizer at 45 degrees with respect to the initial polarization.

Question 5)

a) This one is a fun question because it shows the same precession we did with the spin in a magnetic field but in the Heisenberg picture. Take the Heisenberg equation, and calculate the right-hand side for each spin operator. Since none of the operators is explicitly time-dependent, we only need the commutator with the Hamiltonian:

$$\frac{dS_x}{dt} = \frac{i}{\hbar}[H, S_x] = -\omega S_y, \tag{44}$$

$$\frac{dS_y}{dt} = \frac{i}{\hbar}[H, S_y] = \omega S_x, \tag{45}$$

$$\frac{dS_z}{dt} = \frac{i}{\hbar}[H, S_z] = 0. \tag{46}$$

where we used $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$ wih $\epsilon_{ijk} = 1$ if (i, j, k) is an even permutation of (x, y, z), -1 if it is an odd permutation, and 0 otherwise.

---

b) Let's differentiate them again. The z one is trivial,

$$\frac{d^2S_z}{dt^2} = 0. \tag{47}$$

For the x and y components, we have

$$\frac{d^2S_x}{dt^2} = -\omega\frac{dS_y}{dt} = -\omega^2 S_x, \tag{48}$$

$$\frac{d^2S_y}{dt^2} = \omega\frac{dS_x}{dt} = -\omega^2 S_y. \tag{49}$$

The solution to the second-order differential equations for S_x and S_y are simple harmonic oscillations:

$$\hat{S}_x(t) = A_x \cos(\omega t) + B_x \sin(\omega t), \tag{50}$$

$$\hat{S}_y(t) = A_y \cos(\omega t) + B_y \sin(\omega t) \tag{51}$$

with constants A_x and B_x determined by the initial conditions of the system in the Heisenberg picture:

$$\hat{S}_x(t=0) = A_x, \quad \hat{S}_y(t=0) = A_y. \tag{52}$$

Using the first-order differential equations, we can relate the constants B_x and B_y to the initial values of S_x and S_y:

$$B_x = \frac{d\hat{S}_x}{dt}\bigg|_{t=0} \frac{1}{\omega} = -\hat{S}_y(0), \quad B_y = \frac{d\hat{S}_y}{dt}\bigg|_{t=0} \frac{1}{\omega} = \hat{S}_x(0). \tag{53}$$

Therefore,

$$\hat{S}_x(t) = \hat{S}_x(0) \cos(\omega t) - \hat{S}_y(0) \sin(\omega t), \tag{54}$$

$$\hat{S}_y(t) = \hat{S}_y(0) \cos(\omega t) + \hat{S}_x(0) \sin(\omega t), \tag{55}$$

$$\hat{S}_z(t) = \hat{S}_z(0). \tag{56}$$

---

c) The key observation is that the Bloch vector, defined as

$$\vec{S}(t) = (\langle \hat{S}_x(t) \rangle, \langle \hat{S}_y(t) \rangle, \langle \hat{S}_z(t) \rangle) \tag{57}$$

is precessing around the z axis with a frequency ω. The operators here are written in terms of the Heisenberg picture, so the expectation value is taken with respect to the initial state of the system. Before, we had used the time-evolved states and calculated their expectation values in the Schrödinger picture. Clearly, the result must be the same.