# HW 7: Simple Harmonic Oscillator

This homework walks you through S&N Section 2.3. The Harmonic Oscillator Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}, \qquad [\hat{x}, \hat{p}] = i\hbar. \tag{1}$$

We define the annihilation operator $a$, the creation operator $a^\dagger$, and the number operator $N$ as

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right), \qquad \hat{N} = \hat{a}^\dagger\hat{a}. \tag{2}$$

The eigenkets of $\hat{N}$ are labeled by their eigenvalue $n$, $\hat{N}\,|n\rangle = n\,|n\rangle$, and are normalized, $\langle n|n\rangle = 1$.

Recall that $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{N} = \hat{a}^\dagger\hat{a}$, such that

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right). \tag{3}$$

such that $\hat{H}\,|n\rangle = E_n\,|n\rangle$ with $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$.

a) Show that $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

b) Show that

$$\hat{N}\left(\hat{a}\,|n\rangle\right) = (n-1)\left(\hat{a}\,|n\rangle\right), \qquad \hat{N}\left(\hat{a}^\dagger\,|n\rangle\right) = (n+1)\left(\hat{a}^\dagger\,|n\rangle\right), \tag{4}$$

In other words, $\hat{a}$ lowers the eigenvalue of $\hat{N}$ by one, and $\hat{a}^\dagger$ raises it by one, justifying the names "annihilation" and "creation" come from.

Note that $n \geq 0$ by writing $n = \langle n|\,\hat{N}\,|n\rangle$ as the squared norm of the ket $\hat{a}\,|n\rangle$. The lowest eigenvalue is $n = 0$ with $\hat{a}\,|0\rangle = 0$ and $n = 0, 1, 2, ...$; the spectrum is discrete.

c) Given that $\hat{a}\,|n\rangle = c\,|n-1\rangle$ for some number $c$. Compute $|c|^2$ by taking the norm of both sides, and choose $c$ real and positive. Do the same for $\hat{a}^\dagger\,|n\rangle$. You should find

$$\hat{a}\,|n\rangle = \sqrt{n}\;|n-1\rangle, \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\;|n+1\rangle. \tag{5}$$

d) Inverting the definitions of $a$ and $a^\dagger$, we get

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right), \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\hat{a} + \hat{a}^\dagger\right). \tag{6}$$

What can you conclude about the expectation value of position for any number state $|n\rangle$ in the harmonic oscillator? Is this always true for superpositions of number states?

e) Compute $\langle n|\,\hat{x}^2\,|n\rangle$ and $\langle n|\,\hat{p}^2\,|n\rangle$. *Hint:* in $(a+a^\dagger)^2$, only the terms with one $a$ and one $a^\dagger$ survive, and $aa^\dagger + a^\dagger a = 2N + 1$. Show that $\Delta x\,\Delta p = \left(n + \frac{1}{2}\right)\hbar$ in the state $|n\rangle$. Which state saturates the uncertainty principle?

f) Now we find the ground-state wave function $\langle x|0\rangle$. Project $a\,|0\rangle = 0$ onto $\langle x|$ and use $\langle x|\,p\,|\alpha\rangle = -i\hbar\frac{d}{dx}\,\langle x|\alpha\rangle$ to obtain the first-order differential equation

$$\left(x + x_0^2\frac{d}{dx}\right)\langle x|0\rangle = 0, \tag{7}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. Find the normalized solution

$$\psi_0(x) = \langle x|0\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}} \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]. \tag{8}$$

Note then that using $|1\rangle = a^\dagger |0\rangle$, we get

$$\langle x|1\rangle = \sqrt{2}\,\frac{x}{x_0}\,\langle x|0\rangle\,. \tag{9}$$

and similar for higher excited states, e.g.,

$$\langle x|2\rangle = \frac{1}{\sqrt{2}}\left(2\frac{x^2}{x_0^2} - 1\right)\langle x|0\rangle\,. \tag{10}$$

g) Finally, the time evolution. Use the Heisenberg equation of motion to show that, in the Heisenberg picture,

$$\frac{da}{dt} = -i\omega a \qquad \Longrightarrow \qquad a(t) = a(0)\,e^{-i\omega t}. \tag{11}$$

and that

$$\hat{x}(t) = \hat{x}(0)\cos\omega t + \frac{\hat{p}(0)}{m\omega}\sin\omega t, \tag{12}$$

which is the classical solution, with operators in place of numbers. Why is $\langle\hat{x}\rangle$ zero at all times even though the operator $\hat{x}(t)$ oscillates in time?

**Bonus)** Let $\hat{x}$ and $\hat{p}$ denote Schrödinger-picture operators. Show that

$$\hat{A}(t) = \hat{x}\cos\omega t - \frac{\hat{p}}{m\omega}\sin\omega t$$

is a conserved observable. For a fixed real parameter $\lambda$, define

$$\hat{W}_\lambda(t) = e^{-i\lambda\hat{A}(t)/\hbar}.$$

Determine $\hat{W}_\lambda^\dagger\,\hat{x}\,\hat{W}_\lambda$ and $\hat{W}_\lambda^\dagger\,\hat{p}\,\hat{W}_\lambda$ to order $\mathcal{O}(\lambda)$.