Question 1) Probability density and current: The time-dependent Schrödinger equation in 3D is

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t) + V(\mathbf{x})\psi(\mathbf{x},t).$$

Using the Schrödinger equation, we can calculate the probability density $\rho(\mathbf{x},t) = |\psi(\mathbf{x},t)|^2$, which gives the probability of finding the particle at an infinitesimally small region around $\mathbf{x}$ at time t. For a closed system, the probability density should satisfy the continuity equation:

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot\mathbf{j}(\mathbf{x},t) = 0. \tag{1}$$

This is a statement of local conservation of probability, with $\mathbf{j}(\mathbf{x},t)$ the probability current density (how much probability is flowing through unit area per unit time at given $\mathbf{x}$ and t).

a) Derive the standard expression for the probability current density:

$$\mathbf{j}(\mathbf{x},t) = \frac{\hbar}{2mi}\left(\psi^*(\mathbf{x},t)\nabla\psi(\mathbf{x},t) - \psi(\mathbf{x},t)\nabla\psi^*(\mathbf{x},t)\right).$$

Use the continuity equation in Eq. (1).

b) Now consider a polar form for the wave function: $\psi(\mathbf{x},t) = \sqrt{\rho(\mathbf{x},t)}e^{iS(\mathbf{x},t)/\hbar}$, where $\rho(\mathbf{x},t)$ is the probability density and $S(\mathbf{x},t)$ is the phase of the wave function. Show that the probability current density can be written as

$$\mathbf{j}(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)}{m}\nabla S(\mathbf{x},t).$$

What does this mean for a *real* wavefunction $\psi(\mathbf{x})$?

c) For a plane-wave solution of the form $\psi(\mathbf{x},t) = \frac{1}{\sqrt{V}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, where V is some normalization volume (formally, $V\to\infty$ for a bona-fide plane wave), $\mathbf{k}$ is the wave vector, and ω is the angular frequency, compute the probability density $\rho(\mathbf{x},t)$. Is it time-dependent? How about the probability current density $\mathbf{j}(\mathbf{x},t)$?

Question 2) The Propagator: The propagator $K(x_1,t_1;x_0,t_0)$ is defined as the transition amplitude between two space-time points, obtained by applying the time-evolution operator $U(t_1,t_0)$ between position eigenstates:

$$K(x_1,t_1;x_0,t_0) = \langle x_1|\,\hat{U}(t_1,t_0)\,|x_0\rangle = \langle x_1|\,e^{-\frac{i}{\hbar}\hat{H}(t_1-t_0)}\,|x_0\rangle\,, \tag{2}$$

with the latter equation valid for a time-independent Hamiltonian $\hat{H}$. We will prove a few properties of the propagator in this exercise.

a) Show that the propagator obeys Schrödinger's equation by differentiating the definition of the propagator with respect to t_1 and using $\partial_{t_1}U(t_1,t_0) = -\frac{i}{\hbar}\hat{H}(t_1)U(t_1,t_0)$.

b) Show that the wavefunction at time t_1 can be expressed in terms of the propagator and the initial wavefunction at time t_0 as

$$\psi(x_1,t_1) = \int dx_0\, K(x_1,t_1;x_0,t_0)\psi(x_0,t_0), \tag{3}$$

where $\psi(x,t) = \langle x|\psi(t)\rangle$.

c) prove the following properties of the propagator:

i) **Initial condition**: At equal times, the propagator reduces to the Dirac delta function

$$K(x_1, t; x_0, t) = \delta(x_1 - x_0). \tag{4}$$

ii) **Composition**: The propagator satisfies the composition property

$$K(x_2, t_2; x_0, t_0) = \int dx_1 \, K(x_2, t_2; x_1, t_1) K(x_1, t_1; x_0, t_0), \tag{5}$$

for any intermediate time t_1.