Homework 9

Please take a look at our notes and Sec. 2.6 of S&N. For more, see also Chapter 8 of Shankar, in particular Sec. 8.6 on quadratic potentials. Sakurai labels the endpoints by x' and x'' but I will keep the notation of our notes, $K(x_1,t_1;x_0,t_0) = \langle x_1|\,\hat{U}(t_1,t_0)\,|x_0\rangle$ and work in one dimension with a time interval $T \equiv t_1 - t_0 > 0$.

Recall Feynman's path integral representation of the propagator,

$$K(x_1,t_1;x_0,t_0) = \int_{x(t_0)=x_0}^{x(t_1)=x_1} \mathcal{D}[x(t)]\, \exp\left(\frac{i}{\hbar}S[x(t)]\right), \qquad S[x(t)] = \int_{t_0}^{t_1} dt\, L(x,\dot{x}), \tag{1}$$

where the integral runs over all paths with fixed endpoints, and the classical path $x_{\rm cl}(t)$ is the one that makes the action stationary, $\delta S = 0$, leading to the Euler–Lagrange equation for $x_{\rm cl}(t)$:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0. \tag{2}$$

The action evaluated on the classical path is denoted by $S_{\rm cl}(x_1,t_1;x_0,t_0) \equiv S[x_{\rm cl}(t)]$. Recall the free-particle propagator:

$$K(x_1,t_1;x_0,t_0) = \sqrt{\frac{m}{2\pi i\hbar T}}\, \exp\left(\frac{im(x_1-x_0)^2}{2\hbar T}\right). \tag{3}$$

a) For the free particle, $L = \frac{1}{2}m\dot{x}^2$, find the classical path $x_{\rm cl}(t)$ with $x_{\rm cl}(t_0) = x_0$ and $x_{\rm cl}(t_1) = x_1$. Show that

$$S_{\rm cl}(x_1,t_1;x_0,t_0) = \frac{m(x_1-x_0)^2}{2T}. \tag{4}$$

This shows that Eq. (3) can be written as $K = \sqrt{m/2\pi i\hbar T}\, e^{iS_{\rm cl}/\hbar}$ and that the sum over all paths in Eq. (1) collapses to a single phase, evaluated on a single path (the classical path), times a prefactor that does not depend on x_0 or x_1. This result is exact for the free particle.

b) Show that the classical action satisfies

$$\frac{\partial S_{\rm cl}}{\partial x_1} = p_{\rm cl}(t_1), \qquad \frac{\partial S_{\rm cl}}{\partial t_1} = -E_{\rm cl}, \tag{5}$$

where $p_{\rm cl} = m\dot{x}_{\rm cl}$ and $E_{\rm cl} = p_{\rm cl}^2/2m$ are the momentum and energy of the classical path. (These are the Hamilton-Jacobi relations from classical mechanics with $S_{\rm cl}$ playing the role of Hamilton's principal function.)

Now consider a Lagrangian with a potential that is at most quadratic in x,

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \qquad V(x) = c_0 + c_1 x + c_2 x^2, \tag{6}$$

where c_0, c_1, and c_2 are constants. This includes the free particle, a particle subject to a uniform force (e.g., gravity, $V(x) = -mgx$), and the harmonic oscillator, for example.

c) Any given path with the specified endpoints can be written as the classical path plus a fluctuation,

$$x(t) = x_{\rm cl}(t) + y(t), \qquad y(t_0) = y(t_1) = 0. \tag{7}$$

Using this decomposition, show that the action for a Lagrangian like in Eq. (6) splits exactly into

$$S[x_{\rm cl} + y] = \int_{t_0}^{t_1} dt\, L(x_{\rm cl} + y, \dot{x}_{\rm cl} + \dot{y}) = S_{\rm cl} + \int_{t_0}^{t_1} dt \left[\frac{1}{2} m\dot{y}^2 - c_2\, y^2\right]. \tag{8}$$

Hint: rewrite $S[x_{\rm cl} + y]$ in terms of y. The terms linear in y can be integrated by parts with the boundary conditions on y in Eq. (7). You will also need the Euler-Lagrange equation for $x_{\rm cl}$.

d) Change integration variables in Eq. (1) from $x(t)$ to $y(t)$ (a constant shift leaves the measure invariant, $\mathcal{D}[x(t)] = \mathcal{D}[y(t)]$) to show that the path integral factorizes into a part depending only on the classical action and a part depending only on the fluctuations:

$$K(x_1, t_1; x_0, t_0) = F(T)\, \exp\left(\frac{i}{\hbar} S_{\rm cl}(x_1, t_1; x_0, t_0)\right), \tag{9}$$

with

$$F(T) = \int_{y(t_0)=0}^{y(t_1)=0} \mathcal{D}[y(t)]\, \exp\left(\frac{i}{\hbar} \int_{t_0}^{t_1} dt \left[\frac{1}{2} m\dot{y}^2 - c_2\, y^2\right]\right). \tag{10}$$

Argue that F depends only on T (and on m, c_2, $\hbar$), but not on the endpoints x_0 and x_1, nor on c_0 or c_1. This is the precise sense in which "only the classical action matters" for quadratic Lagrangians: all of the dependence on where the particle starts and ends is in $e^{iS_{\rm cl}/\hbar}$.

e) Now suppose $V(x)$ contains an x^4 term. Which new terms appear in Eq. (8), and why does the argument in part (d) break down? In that case, $e^{iS_{\rm cl}/\hbar}$ is only one part of the full answer, but it becomes dominant in semiclassical approximation (for example, taking$\hbar \to 0$).



For a generic potential, the paths that dominate Eq. (1) as $\hbar \to 0$ are those near $x_{\rm cl}(t)$, because the phase $e^{iS/\hbar}$ oscillates wildly everywhere else and neighboring paths cancel. This is the stationary-phase approximation and the path-integral version of what is called the WKB approximation (we will cover approximation methods next semester). It shows how classical mechanics emerges from quantum mechanics. Quadratic Lagrangians are special because the stationary-phase approximation is exact. The same fact explains why quadratic quantum field theories (usually called "free" or "non-interacting" theories) can be solved exactly, while interacting ones typically require approximate methods such as perturbation theory.