Review Notes

Quantum systems and Hamiltonians

Free particle: $\hat{H} = \frac{\hat{p}^2}{2m}$ → kinetic energy only.

+ **Potential** $V(\hat{x})$: $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

Rewrite it as a differential operator on the wavefunction:

$\psi(x) = \langle x|\psi\rangle$

- Schrodinger's eq: $i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle$

In position basis:

$$\langle x|i\hbar\partial_t|\psi\rangle = \langle x|\hat{H}|\psi\rangle$$

$$\Rightarrow i\hbar\partial_t\psi(x) = \frac{\langle x|\hat{p}^2|\psi\rangle}{2m} + \langle x|V(\hat{x})|\psi\rangle$$

Note: $V(\hat{x})|x\rangle = V(x)|x\rangle \Rightarrow \langle x|V(\hat{x}) = \langle x|V(x)$

by hermiticity of hamiltonian,

Also $\langle x|\hat{p}|\psi\rangle = -i\hbar\partial_x\psi(x)$

$$\Rightarrow \langle x|\hat{p}^2|\psi\rangle = -\hbar^2\partial_x^2\psi(x)$$

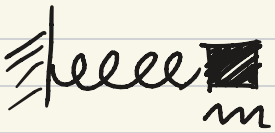
So:

$$i\hbar\partial_t|\psi\rangle = \hat{H}|\psi\rangle \Rightarrow \boxed{i\hbar\partial_t\psi(x) = \left(\frac{-\hbar^2\partial_x^2}{2m} + V(x)\right)\psi(x)} \;//$$

(?) Calculate the probability density and current for a Gaussian wavepacket:

$$\psi(x) = \left(\frac{1}{2\pi\sigma_x^2}\right)^{1/4}\exp\left[-\frac{(x-x_0)^2}{2\sigma_x^2} + ikx\right] \longrightarrow \rho(x,t);\ J(x,t).$$

How do we get the simple harmonic oscillator?



$F = -Kx$; $F = -\partial_x V$

recall frequency of oscillation in classical problem is $\omega = \sqrt{\frac{K}{m}}$.

$$V = \int^x dx'(Kx') = \frac{m\omega^2 x^2}{2}$$

So it makes sense that:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{\hbar\omega}{2}\left(\frac{\hat{x}}{x_0}\right)^2$$

w/ $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ → Typical distance scale.

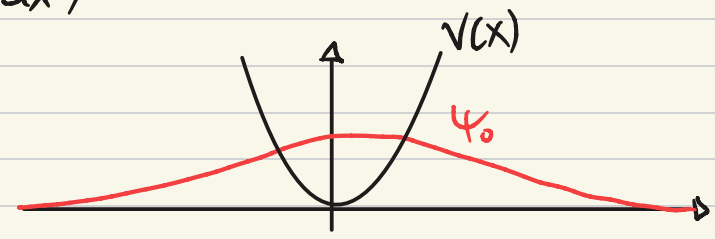
Recall $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$; $[\hat{a}, \hat{a}^\dagger] = 1$

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) ; \quad \hat{N} = \hat{a}^\dagger\hat{a} ; \quad \hat{N}|n\rangle = n|n\rangle$$

$$\hat{H}|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle \rightarrow \text{evenly spaced}$$

Solutions $\hat{a}|0\rangle = 0 \Rightarrow \langle x|\hat{a}|0\rangle = 0$

$$\left(x + x_0^2\frac{d}{dx}\right)\psi_0(x) = 0 \Rightarrow \psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-x^2/2x_0^2}.$$



(?) find other wave functions, just do:

$$\psi_1(x) = \langle x|1\rangle = \langle x|a^\dagger|0\rangle = D(x)\,\psi_0(x)$$

↳ differential op.

(?) Find the energy levels of a harmonic oscillator with a linear term:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + \lambda\hat{x}$$

This corresponds to an oscillator subject to an uniform force: $F = -\lambda \Rightarrow V = \int^x dx' F = \lambda x$.

* e.g. a charged oscillator in a uniform field.

Rotations:

Definition of a vector: $\hat{D}^\dagger(R)\,\hat{V}_i\,\hat{D}(R) = R_{ij}\,\hat{V}_j$

equivalent to $[\hat{V}_i, \hat{J}_j] = i\hbar\,\varepsilon_{ijk}\,\hat{V}_k$

↳ generator of rotations

(?) Consider the following Hamiltonian

$$\hat{H} = \frac{\hat{J}^2}{2I} \quad \text{with} \quad \hat{J}^2 = J_x^2 + J_y^2 + J_z^2$$

I → Moment of inertia.

Show that the angular momentum in any direction is conserved:

$$\frac{d}{dt}(\hat{J}\cdot\hat{n}) = \hat{n}\cdot\left(\frac{d\vec{J}}{dt}\right) = \hat{n}\cdot\left(\frac{[\vec{J}, \hat{H}]}{i\hbar} + \frac{\partial \vec{J}}{\partial t}\right) = 0$$

by virtue of $[J^2, J_i] = 0$ for $i = x, y, z$.

What symmetry does this correspond to?

Rotational invariance: $[\hat{D}(R), \hat{V}] = 0$:

$$\hat{D}^\dagger(R)\,\hat{V}_i\,\hat{D}(R) = R_{ij}\,\hat{V}_j \Rightarrow \boxed{\hat{V}_i = R_{ij}\,\hat{V}_j}$$

For any vector observable $\langle \hat{V}_i \rangle = R_{ij} \langle \hat{V}_j \rangle$ //

Equivalently:

$[\hat{V}_i, \hat{J}_j] = 0 \Rightarrow$ Rotational invariance around axis j

$\Downarrow$

Share a set of eigenvectors.

Recall: $\hat{D}(\hat{n}, \theta) = e^{-\frac{i}{\hbar}(\hat{n}\cdot\hat{\vec{J}})\theta}$