Homework 6: The Rotating Wave Approximation

In the lecture notes we studied Rabi oscillations driven by a circularly polarized field, for which the problem could be solved exactly. In practice, however, the driving field is often *linearly* polarized. In this exercise you will show that the linearly polarized problem reduces to the circularly polarized one after an approximation known as the *rotating wave approximation* (RWA), and you will estimate the leading correction due to the terms that are dropped. Throughout, we work in natural units with $\hbar = 1$ and use the free Hamiltonian

$$\hat{H}_0 = \frac{\omega_0}{2}\sigma_z, \tag{1}$$

with eigenstates $|+\rangle$ (energy $+\omega_0/2$) and $|-\rangle$ (energy $-\omega_0/2$), and a *linearly polarized* perturbation

$$\hat{V}(t) = \Omega\cos(\omega t)\,\sigma_x, \tag{2}$$

where $\Omega \ll \omega_0$ and the driving frequency ω is near resonance, $|\omega - \omega_0| \ll \omega_0$.

(a) Decomposition into rotating components. Using $\sigma_x = \sigma_+ + \sigma_-$ with $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$, show that $\hat{V}(t)$ can be written as

$$\hat{V}(t) = \hat{V}^{\mathrm{co}}(t) + \hat{V}^{\mathrm{ctr}}(t), \tag{3}$$

where

$$\hat{V}^{\mathrm{co}}(t) = \frac{\Omega}{2}(\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}), \qquad \hat{V}^{\mathrm{ctr}}(t) = \frac{\Omega}{2}(\sigma_+ e^{+i\omega t} + \sigma_- e^{-i\omega t}). \tag{4}$$

Compare $\hat{V}^{\mathrm{co}}(t)$ with the perturbation used in the lecture notes (Eq. 3 of the Rabi notes). What is the physical interpretation of these two terms?
(*Hint:* Consider which component of a classical magnetic field rotates in the same sense as the Larmor precession.)

(b) Interaction picture. Go to the interaction picture. Using $e^{i\hat{H}_0 t}\,\sigma_\pm\,e^{-i\hat{H}_0 t} = \sigma_\pm\,e^{\pm i\omega_0 t}$, show that

$$\hat{V}_I(t) = \frac{\Omega}{2}\begin{pmatrix} 0 & e^{-i\delta t} + e^{i(\omega_0+\omega)t} \\ e^{i\delta t} + e^{-i(\omega_0+\omega)t} & 0 \end{pmatrix}, \tag{5}$$

where $\delta = \omega - \omega_0$ is the detuning. Identify which terms in this matrix originate from $\hat{V}^{\mathrm{co}}$ and which from $\hat{V}^{\mathrm{ctr}}$. Near resonance ($\delta \approx 0$), argue that the co-rotating terms vary slowly (at frequency $|\delta|$) while the counter-rotating terms oscillate rapidly (at frequency $\omega_0 + \omega \approx 2\omega_0$).

(c) The rotating wave approximation. The RWA consists of dropping the rapidly oscillating counter-rotating terms, i.e., replacing $\hat{V}_I(t) \to \hat{V}_I^{\mathrm{RWA}}(t)$ with

$$\hat{V}_I^{\mathrm{RWA}}(t) = \frac{\Omega}{2}\begin{pmatrix} 0 & e^{-i\delta t} \\ e^{i\delta t} & 0 \end{pmatrix}. \tag{6}$$

Observe that this is identical to the interaction picture potential from the circularly polarized Rabi problem (Eq. 6 of the lecture notes). Conclude that, within the RWA, the transition probability for a linearly polarized driver is given by

$$P^{\mathrm{RWA}}_{+\to-}(t) = \frac{\Omega^2}{\delta^2 + \Omega^2}\,\sin^2\left(\frac{\sqrt{\delta^2+\Omega^2}\,t}{2}\right). \tag{7}$$

Under what conditions is the RWA expected to be a good approximation? Express your answer as an inequality involving Ω, ω, and ω_0.

(d) First-order correction from the counter-rotating terms. Let us now estimate the error introduced by the RWA using first-order perturbation theory. Compute the first-order transition amplitude $|+\rangle \to |-\rangle$ arising from the counter-rotating part alone:

$$A^{\text{ctr}}_{+\to-}(t) = -i\int_0^t dt' \ \langle -|\, \hat{V}_I^{\text{ctr}}(t')\, |+\rangle \,. \tag{8}$$

Show that, near resonance, their ratio scales as

$$\frac{|A^{\text{ctr}}|}{|A^{\text{co}}|} \sim \frac{|\delta|}{\omega_0+\omega} \approx \frac{|\delta|}{2\omega_0}, \tag{9}$$

confirming that the counter-rotating contribution is negligible when $|\delta| \ll \omega_0$.

This question is completely optional but will count 1 full point towards your exam score.

(Bonus) The Bloch-Siegert shift Although the counter-rotating terms do not contribute to the $|+\rangle \to |-\rangle$ transition at first order in any significant way, they *do* affect the diagonal matrix elements at second order, producing a shift in the effective resonance condition. This is called the Bloch-Siegert shift.

Consider the second-order correction to the coefficient of the initial state $|+\rangle$. Using the Dyson series, this is

$$c_+^{(2)}(t) = (-i)^2 \int_0^t dt' \int_0^{t'} dt'' \ \langle +|\, \hat{V}_I^{\text{ctr}}(t')\hat{V}_I^{\text{ctr}}(t'')\, |+\rangle \,. \tag{10}$$

(We use $\hat{V}_I^{\text{ctr}}$ alone, since the cross terms between co- and counter-rotating pieces and the purely co-rotating contribution are already accounted for in the RWA solution.)

(i) Show that $\langle +|\, \hat{V}_I^{\text{ctr}}(t')\hat{V}_I^{\text{ctr}}(t'')\, |+\rangle = \frac{\Omega^2}{4}\, e^{i(\omega_0+\omega)(t'-t'')}$.

(*Hint:* The product $\hat{V}_I^{\text{ctr}}(t')\hat{V}_I^{\text{ctr}}(t'')$ has a diagonal part proportional to $\sigma_+\sigma_- + \sigma_-\sigma_+$. Only one of these contributes when sandwiched between $\langle +| \cdots |+\rangle$.)

(ii) Perform the double time integral and show that, for $(\omega_0+\omega)t \gg 1$, the dominant contribution takes the form

$$c_+^{(2)}(t) \approx -\frac{i\,\Omega^2}{4(\omega_0+\omega)}\, t. \tag{11}$$

(*Hint:* After performing the t'' integral, you will encounter a term that grows linearly in t and oscillatory terms that remain bounded. Keep only the secular (linearly growing) piece.)

(iii) Interpret this result. Argue that the counter-rotating terms effectively shift the energy of $|+\rangle$ by $\Delta E_+ = +\frac{\Omega^2}{4(\omega_0+\omega)}$. By a similar calculation (or by symmetry), argue that $|-\rangle$ is shifted by $\Delta E_- = -\frac{\Omega^2}{4(\omega_0+\omega)}$.

(iv) Conclude that the resonance condition is shifted from $\omega = \omega_0$ to

$$\omega = \omega_0 + \frac{\Omega^2}{2(\omega_0+\omega)} \approx \omega_0 + \frac{\Omega^2}{4\omega_0}, \tag{12}$$

where the last approximation holds near resonance. This is the Bloch-Siegert shift. Estimate its magnitude relative to ω_0 for typical atomic physics parameters ($\omega_0 \sim 10^{15}$ rad/s, $\Omega \sim 10^9$ rad/s) and comment on when it might become experimentally relevant.