Sterile neutrinos with relatively small masses can have interesting phenomenological implications. In this exercise, we will explore the possibility of eV-scale sterile neutrinos and their mixing with active neutrinos. Consider a simple extension of the Standard Model that includes one sterile neutrino ν_s in addition to the three active neutrinos ν_e, ν_μ, and ν_τ. Assume that the sterile neutrino mixes with the active neutrinos through a mixing matrix U:

$$|\nu_\alpha\rangle = \sum_{i=1}^{4} U^*_{\alpha i} |\nu_i\rangle . \tag{0.1}$$

Here, $\alpha = e, \mu, \tau, s$ and $i = 1, 2, 3, 4$.

1. Taking $\Delta m^2_{41} \gg \Delta m^2_{31}, \Delta m^2_{21}$, derive the expression for the oscillation probability $P(\nu_\alpha \to \nu_\beta)$ in vacuum, where $\alpha, \beta = e, \mu, \tau$. This should depend exclusively on $|U_{\alpha 4}|$, $|U_{\beta 4}|$, and Δm^2_{41}.

2. Write the formulae for $P(\nu_e \to \nu_e)$ and $P(\nu_\mu \to \nu_\mu)$ in this limit, as well as for $P(\nu_\mu \to \nu_e)$.

3. If $\Delta m^2_{41} \sim 1$ eV^2, calculate the distance of the first oscillation maximum for neutrinos with an energy of 30 MeV (from μ^+ decay at rest, for example) and 1 GeV (from pion decay in flight). What happens when L is much larger than this distance (recall question 1).

4. Can you relate the disappearance probabilities to the appearance one? What does an appearance signal imply for the disappearance channels?

5. Neutrino sources are rarely pure in terms of flavor content. Take, for example, the Booster neutrino beam at Fermilab, which has a small contamination of 5% ν_e in a primarily ν_μ beam. How does this contamination affect the sensitivity of an experiment searching for $\nu_\mu \to \nu_e$ oscillations due to eV-scale sterile neutrinos? Assume you can only really trust the measurement of the ratio N_{ν_e}/N_{ν_μ}.

Solutions This question is getting us to think about a topic related to a longstanding puzzle in neutrino physics: eV sterile neutrinos. No one would have ever cared as much as about these specific sterile neutrinos if it was not the fact that several neutrino detectors claimed an anomalous oscillation at energies and baselines that corresponded to $\Delta m^2 \sim 1$ eV^2. Note that this is significantly larger than $\Delta m^2_{31} \sim 2 \times 10^{-3}$ eV^2 and $\Delta m^2_{21} \sim 7 \times 10^{-5}$ eV^2, but not *that* much larger. As far as sterile neutrinos go, this was a pretty surprising mass scale to find one at.

Let us start with the general form for the oscillation probability in vacuum:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2\left(\frac{\Delta_{ij}}{2}\right) \tag{0.2}$$

$$+ 2\sum_{i>j} \mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\left(\Delta_{ij}\right), \tag{0.3}$$

where $\Delta_{ij} = \frac{\Delta m^2_{ij}L}{2E}$. One can separate the terms involving the fourth mass eigenstate from those involving only the first three. Since we are taking the limit $\Delta m^2_{41} \gg \Delta m^2_{31}, \Delta m^2_{21}$, the oscillation terms involving only the first three mass eigenstates will be negligible. For simplicity, let us set all the 31 and 21 oscillation terms to zero. We then have

$$\Delta_{31} = \Delta_{21} = \Delta_{32} = 0 \implies \Delta_{41} = \Delta_{42} = \Delta_{43} = \frac{\Delta m^2_{41}L}{2E}. \tag{0.4}$$

So, inside each of the double-sums over $i > j$, we end up with $i = 4$ terms only, and sum j from 1 to 3. The oscillation probability then reduces to

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i=1}^{3} \mathrm{Re}(U^*_{\alpha 4}U_{\beta 4}U_{\alpha i}U^*_{\beta i})\sin^2\left(\frac{\Delta m^2_{41}L}{4E}\right) \tag{0.5}$$

$$+ 2\sum_{i=1}^{3} \mathrm{Im}(U^*_{\alpha 4}U_{\beta 4}U_{\alpha i}U^*_{\beta i})\sin\left(\frac{\Delta m^2_{41}L}{2E}\right). \tag{0.6}$$

Recall from the unitarity of the mixing matrix that:

$$\sum_{i=1}^{4} U_{\alpha i}U^*_{\beta i} = \delta_{\alpha\beta} \implies \sum_{i=1}^{3} U_{\alpha i}U^*_{\beta i} = \delta_{\alpha\beta} - U_{\alpha 4}U^*_{\beta 4}. \tag{0.7}$$

Pulling the sums inside the real and imaginary parts, we find

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\,\mathrm{Re}\left(U^*_{\alpha 4}U_{\beta 4}\sum_{i=1}^{3} U_{\alpha i}U^*_{\beta i}\right)\sin^2\left(\frac{\Delta m^2_{41}L}{4E}\right) \tag{0.8}$$

$$+ 2\,\mathrm{Im}\left(U^*_{\alpha 4}U_{\beta 4}\sum_{i=1}^{3} U_{\alpha i}U^*_{\beta i}\right)\sin\left(\frac{\Delta m^2_{41}L}{2E}\right). \tag{0.9}$$

$$= \delta_{\alpha\beta} - 4\,\mathrm{Re}\left(U^*_{\alpha 4}U_{\beta 4}(\delta_{\alpha\beta} - U_{\alpha 4}U^*_{\beta 4})\right)\sin^2\left(\frac{\Delta m^2_{41}L}{4E}\right) \tag{0.10}$$

$$+ 2\,\mathrm{Im}\left(U^*_{\alpha 4}U_{\beta 4}(\delta_{\alpha\beta} - U_{\alpha 4}U^*_{\beta 4})\right)\sin\left(\frac{\Delta m^2_{41}L}{2E}\right), \tag{0.11}$$

and after using the fact that the imaginary part of the absolute value of the mixing matrix elements is zero, we arrive at

$$\boxed{P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\left(\delta_{\alpha\beta}|U_{\alpha 4}|^2 - |U_{\alpha 4}|^2|U_{\beta 4}|^2\right)\sin^2\left(\frac{\Delta m^2_{41}L}{4E}\right).} \tag{0.12}$$

Therefore, in the limit $\Delta m^2_{41} \gg \Delta m^2_{31}, \Delta m^2_{21}$, the oscillations problem reduces to an effective two-flavor oscillation problem between the active flavors and the sterile one. This is called the **3+1 model** scenario in the literature: 3 active neutrinos + 1 sterile neutrino, with the large mass splitting Δm^2_{41} driving the oscillations between them. As long as the standard oscillations have not yet developed ($\Delta_{31} \ll 1$ and $\Delta_{21} \ll 1$), the oscillation probabilities depend only on the mixing matrix elements $|U_{\alpha 4}|$ and $|U_{\beta 4}|$, as well as on Δm^2_{41}. Let us collect our results. The disappearance probabilities are given by

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - 4|U_{e4}|^2(1-|U_{e4}|^2)\sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right), \\ P(\nu_\mu \to \nu_\mu) &= 1 - 4|U_{\mu 4}|^2(1-|U_{\mu 4}|^2)\sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right), \end{aligned} \tag{0.13}$$

while the appearance probability is

$$P(\nu_\mu \to \nu_e) = 4|U_{\mu 4}|^2|U_{e4}|^2\sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right). \tag{0.14}$$

You will also see the appearance probability written in the following way in the literature:

$$P(\nu_\mu \to \nu_e) = \sin^2(2\theta_{\mu e})\sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right), \tag{0.15}$$

where $\sin^2(2\theta_{\mu e}) = 4|U_{\mu 4}|^2|U_{e4}|^2$, and similarly for the disappearance channels,

$$\begin{aligned} P(\nu_e \to \nu_e) &= 1 - \sin^2(2\theta_{ee})\sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right), \\ P(\nu_\mu \to \nu_\mu) &= 1 - \sin^2(2\theta_{\mu\mu})\sin^2\left(\frac{\Delta m^2_{41} L}{4E}\right), \end{aligned} \tag{0.16}$$

where $\sin^2(2\theta_{ee}) = 4|U_{e4}|^2(1-|U_{e4}|^2)$ and $\sin^2(2\theta_{\mu\mu}) = 4|U_{\mu 4}|^2(1-|U_{\mu 4}|^2)$. These are just different ways of parameterizing the same physics, but the mixing angle notation is often more physically intuitive and maintains the nice property that apperance probabilities depend on the same parameters as the disappearance probabilities.

One can show that as long as the new mixing elements $|U_{e4}|^2$ and $|U_{\mu 4}|^2$ are small, the apperance and disapperance angles are related by

$$\sin^2(2\theta_{\mu e}) \simeq \frac{1}{4}\sin^2(2\theta_{ee})\sin^2(2\theta_{\mu\mu}), \tag{0.17}$$

where we dropped terms of order $|U_{\alpha 4}|^4$. This relation is useful when comparing results from different experiments that may have access to different oscillation channels. In more generality, you can also show that

$$\sin^2(2\theta_{\mu e}) = \left(1 - \sqrt{1-\sin^2(2\theta_{ee})}\right)\left(1 - \sqrt{1-\sin^2(2\theta_{\mu\mu})}\right). \tag{0.18}$$

The distance of the first oscillation maximum can be found by setting the argument of the sine squared to $\frac{\pi}{2}$:

$$\frac{\Delta m_{41}^2 L_{\text{osc}}}{4E} = 1.27 \frac{\Delta m_{41}^2[\text{eV}^2] L_{\text{osc}}[\text{km}]}{E[\text{GeV}]} \stackrel{!}{=} \frac{\pi}{2} \implies L_{\text{osc}}[\text{km}] = \frac{1.57\pi E[\text{GeV}]}{\Delta m_{41}^2[\text{eV}^2]}. \tag{0.19}$$

For $E = 30$ MeV and $\Delta m_{41}^2 = 1$ eV^2, we find

$$L_{\text{max}} = 150 \text{ m}. \tag{0.20}$$

For $E = 1$ GeV, we find

$$L_{\text{max}} = 5 \text{ km}. \tag{0.21}$$

When L is much larger than this distance, the oscillation probability averages out to

$$\langle P(\nu_\alpha \to \nu_\beta) \rangle = 2|U_{\alpha 4}|^2 |U_{\beta 4}|^2. \tag{0.22}$$

and we can relate the disappearance probabilities to the appearance one as follows:

$$P(\nu_\mu \to \nu_e) \simeq \frac{1}{4} P(\nu_e \to \nu_e) P(\nu_\mu \to \nu_\mu) \tag{0.23}$$

> An appearance signal in $\nu_\mu \to \nu_e$ implies that there must be corresponding disappearance signals in both the ν_e and ν_μ channels. For apperance, both $|U_{e4}|^2$ and $|U_{\mu 4}|^2$ must be non-zero, which in turn implies that there must be some level of disappearance in both channels.

Say we can only measure the ratio N_{ν_e}/N_{ν_μ}. Without oscillations, this ratio is simply

$$x \equiv \frac{N_{\nu_e}^0}{N_{\nu_\mu}^0} = 5\%. \tag{0.24}$$

In the presence of sterile neutrino oscillations, this ratio becomes

$$\frac{N_{\nu_e}}{N_{\nu_\mu}} = \frac{N_{\nu_\mu}^0 P(\nu_\mu \to \nu_e) + N_{\nu_e}^0 P(\nu_e \to \nu_e)}{N_{\nu_\mu}^0 P(\nu_\mu \to \nu_\mu) + N_{\nu_e}^0 P(\nu_e \to \nu_\mu)} = \frac{P(\nu_\mu \to \nu_e) + xP(\nu_e \to \nu_e)}{P(\nu_\mu \to \nu_\mu) + xP(\nu_e \to \nu_\mu)}. \tag{0.25}$$

Because the appearance probability $P(\nu_e \to \nu_\mu) = P(\nu_\mu \to \nu_e)$ is typically very small, we can neglect it in the denominator since it shows up multiplied by the small contamination fraction x. We then have:

$$\frac{N_{\nu_e}}{N_{\nu_\mu}} \simeq \frac{P(\nu_\mu \to \nu_e) + xP(\nu_e \to \nu_e)}{P(\nu_\mu \to \nu_\mu)}. \tag{0.26}$$

The contamination of ν_e in the beam adds a term proportional to $P(\nu_e \to \nu_e)$ in the numerator, which can mask the appearance signal $P(\nu_\mu \to \nu_e)$. This makes it more challenging to isolate and identify the effects of sterile neutrino oscillations, as the observed ratio now depends on both appearance and disappearance probabilities. This is a degeneracy that has plagued several experimental searches for eV-scale sterile neutrinos.

Let us find the point where the oscillated ratio equals the unoscillated one, despite the presence of sterile neutrino oscillations. Setting $N_{\nu_e}/N_{\nu_\mu} = N^0_{\nu_e}/N^0_{\nu_\mu}$, we have

$$\frac{P(\nu_\mu \to \nu_e) + xP(\nu_e \to \nu_e)}{P(\nu_\mu \to \nu_\mu)} = x, \tag{0.27}$$

which can be rearranged to

$$P(\nu_\mu \to \nu_e) = x(P(\nu_\mu \to \nu_\mu) - P(\nu_e \to \nu_e)). \tag{0.28}$$

This equation tells us that for a given value of x, there is a combination of appearance and disappearance probabilities that will make the oscillated ratio indistinguishable from the unoscillated one. When $P(\nu_\mu \to \nu_\mu) = P(\nu_e \to \nu_e)$, the RHS is zero, and the appearance probability can take any value and still satisfy the equation. This is easy to understand: if both disappearance probabilities are equal, the contamination effect cancels out, and the ratio remains unchanged regardless of the appearance probability.

In terms of the mixing matrix elements, this condition can be expressed as

$$|U_{\mu4}|^2|U_{e4}|^2 = x\left(|U_{e4}|^2 - |U_{\mu4}|^2\right) \implies |U_{\mu4}|^2 = \frac{x}{1 - \frac{x}{|U_{e4}|^2}}. \tag{0.29}$$

This relation defines a curve in the $|U_{\mu4}|^2$ vs. $|U_{e4}|^2$ parameter space along which the oscillated ratio equals the unoscillated one, highlighting the degeneracy introduced by the ν_e contamination in the beam.

If $|U_{\mu4}|^2 > |U_{e4}|^2$, the RHS is negative, which is unphysical. This signals that for $|U_{\mu4}|^2$ significantly larger than $|U_{e4}|^2$, the contamination effect cannot mask the appearance signal, and the oscillated ratio will differ from the unoscillated one.