1 Basics of Early Universe Spacetime Geometry

Now we will explore some aspects of early universe cosmology. Because cosmology deals with the entire Universe and because according to Big Bang cosmology the Universe was once much smaller and full of stuff, we will need to work in the context of General Relativity (GR). If you have never seen GR before, do not worry: we will only need a few basic concepts to get started. In fact, all we really need is the FLRW metric, which we will discuss in these notes. Our discussion on dark matter will then proceed from there.

The Minkowski metric and line element. Let me start with flat Minkowski spacetime in 3 + 1 dimensions: $x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$, where $x^0 = t$ is the time coordinate and x^i are the spatial coordinates. The metric is given by the Minkowski metric, a rank 2 tensor defined as:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.1}$$

which has the signature $(+, -, -, -)$ we are used to. A line element in an arbitrary spacetime with a metric tensor $g_{\mu\nu}$ at location r^μ can be written as

$$ds^2 = dr_\mu dr_\nu \eta^{\mu\nu} = \left(\frac{\partial r^\mu}{\partial x^\alpha} dx^\alpha\right) \left(\frac{\partial r^\nu}{\partial x^\beta} dx^\beta\right) \eta_{\mu\nu} = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{1.2}$$

In Minkowski spacetime, we have $g_{\mu\nu} = \eta_{\mu\nu}$, and

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{1.3}$$

This is just some generalization of the Pythagorean theorem to spacetime intervals.

Spherical coordinates. To desmistify the FLRW metric, let me derive the Minkowski metric in spherical coordinates, (t, r, θ, ϕ). Starting from Cartesian coordinates

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta,$$

we get the differentials

$$\begin{aligned} dx &= \sin\theta \cos\phi \, dr + r\cos\theta\cos\phi \, d\theta - r\sin\theta \sin\phi \, d\phi, \\ dy &= \sin\theta \sin\phi \, dr + r\cos\theta \sin\phi \, d\theta + r \sin\theta \cos\phi \, d\phi, \\ dz &= \cos\theta \, dr - r\sin\theta \, d\theta. \end{aligned}$$

In Eq. (1.3), when you expand and sum the squares, all mixed terms proportional to $dr\, d\theta$ and $dr\, d\phi$ cancel, and the remaining terms combine to give

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) = dr^2 + r^2 d\Omega^2.$$

where we defined $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, which is just the line element on the unit sphere. Therefore the Minkowski line element is just

$$\boxed{ds^2 = dt^2 - \left(dr^2 + r^2 d\Omega^2\right) \quad \text{Minkowski metric in spherical coordinates.}}$$

The definition of the metric is easy to read off from this line element:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} .$$

with $\mu, \nu = 0, 1, 2, 3$ corresponding to t, r, θ, ϕ, respectively.

FLRW metric. The presence of matter modifies the metric: matter tells spacetime how to curve. For an isotropic and homogeneous Universe, the energy density and pressure of the background matter (literally all the stuff in the Universe) modify spacetime in a very simple and elegant way. It can be shown that the most general metric that describes such a Universe is given by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, for which a line element is given by

$$\boxed{ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \quad \text{(FLRW metric)},} \tag{1.4}$$

where $a(t)$ is called the scale factor. It scales the spatial part of the metric equally for different times t. The Universe is isotropic and homogeneous, so it is unsurprising that the modifications to the metric show up as universal scalings laws: if it were any different, we would be able to identify preferred directions or locations in space, which at the largest scales is not the case. You may ask if this is a good model for the Universe, and the answer is yes: observations of the cosmic microwave background (CMB) and large scale structure (LSS) confirm that the Universe is isotropic and homogeneous to a very good approximation at large scales of $\gtrsim 100$ Mpc (1 kpc $= 3.086 \times 10^{19}$ m is a typical galactical scale while 1 Mpc $= 10^3$ kpc is a typical scale for clusters of galaxies).

The constant k determines the curvature of space. Clearly, for $k = 0$, we recover the Minkowski metric in spherical coordinates derived above up to the scale factor $a(t)$, which is just $a(t) = 1$ in vacuum. This is the flat Universe case.

Closed spherical geometry. Let us see how something like $k \neq 0$ can arise. Below we will show how a closed spherical universe corresponds to $k = +1$. In that spirit, let us consider a 3-dimensional manifold embedded in some 4-dimensional space. In particular, consider the 3-dimensional sphere S^3 of radius ρ embedded in $\mathbb{R}^4$. Using hyper-spherical coordinates $(\rho, \psi, \theta, \phi)$, where $\psi, \theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, the Cartesian coordinates in $\mathbb{R}^4$ are given by

$$\begin{aligned} x^1 &= \rho \sin\psi \sin\theta \cos\phi, \\ x^2 &= \rho \sin\psi \sin\theta \sin\phi, \\ x^3 &= \rho \sin\psi \cos\theta, \\ x^4 &= \rho \cos\psi, \end{aligned} \tag{1.5}$$

Through a procedure similar to the one above, one can show that the line element in $\mathbb{R}^4$ is

$$ds^2 = d\rho^2 + \rho^2 \left(d\psi^2 + \sin^2\psi \, d\Omega^2 \right) \quad (\mathbb{R}^4), \tag{1.6}$$

where again we used $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. We are interested instead in the geometry of the 3-sphere itself. Therefore, to get the line element on S^3, we fix the radius to some constant value $\rho = R$ to get

$$ds^2 = R^2\left(d\psi^2 + \sin^2\psi\, d\Omega^2\right) \quad (S^3) \tag{1.7}$$

Now perform the coordinate change $r = \sin\psi$ with $(0 \le r \le 1)$. Differentiating gives $dr = \cos\psi\, d\psi$ and using $\cos^2\psi = 1 - \sin^2\psi = 1 - r^2$ we obtain

$$d\psi^2 = \frac{dr^2}{1 - r^2}. \tag{1.8}$$

Substituting into the 3-sphere metric yields

$$ds^2 = R^2\left(\frac{dr^2}{1 - r^2} + r^2 d\Omega^2\right). \tag{1.9}$$

This matches the spatial part of the FLRW metric(!) with $k = +1$ upon identifying the curvature radius of the 3-sphere with the scale factor, $a(t) = R$. Thus a closed spherical spatial geometry corresponds to $k = 1$.

Open hyperbolic geometry. To see how something like $k = -1$ can arise, consider the analogous 3-dimensional hyperbolic space H^3 embedded in $\mathbb{R}^4$. You can obtain hyperbolic coordinates (χ', θ', ϕ') through the simple coordinate transformation of $\psi' = i\psi$, $\theta' = \theta$, $\phi' = \phi$. Because $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$, you can convince yourself that this describes a hyperbolic space by noting that the equation for the 3-sphere, $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = R^2$, becomes the equation for a hyperboloid, $(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 = -R^2$, under this transformation. The line element on H^3 becomes

$$ds^2 = R^2\left(d\psi^2 + \sinh^2\psi\, d\Omega^2\right). \tag{1.10}$$

Now, similarly to before, perform the coordinate change $r = \sinh\psi$ with $(0 \le r < \infty)$. Differentiating gives $dr = \cosh\psi\, d\psi$ and using $\cosh^2\psi = \sinh^2\psi + 1 = r^2 + 1$ we obtain

$$d\psi^2 = \frac{dr^2}{1 + r^2}. \tag{1.11}$$

Substituting into the hyperbolic space metric yields

$$ds^2 = R^2\left(\frac{dr^2}{1 + r^2} + r^2 d\Omega^2\right). \tag{1.12}$$

This matches the spatial part of the FLRW metric with $k = -1$ upon identifying the curvature radius of the hyperbolic space with the scale factor, $a(t) = R$.

Now, the intermediate cases of closed $k > 0$ and an open $k < 0$ Universes are just deformations of spheres and hyperboloids. The key difference is that in closed space-times, if you travel in a straight line long enough, you will eventually return to your starting point. In an open Universe, two parallel lines will eventually diverge from each other as you travel along them.

Note how the radius of our 3-sphere or hyperboloid became the scale factor $a(t)$ in the FLRW metric. This gives another intuitive meaning to the scale factor: it *scales* the distances in our spatial manifold as a function of time. In the S^3 case, for instance, you would say that if $a(t)$ increases with time, then the radius of the 3-sphere is increasing with time, meaning that distances between two points on the sphere are also increasing with time. In other words, the Universe is expanding. As we will see, this happens for our own Universe, which is very well described by the FLRW metric with $k = 0$ and an increasing scale factor $a(t)$, but in an *accelerated* way!

In summary, the FLRW metric describes an isotropic and homogeneous Universe with three possible spatial geometries:

- Flat space: $k = 0$, spatial geometry is Euclidean.
- Closed space: $k > 0$, spatial geometry is spherical.
- Open space: $k < 0$, spatial geometry is hyperbolic.

Observations indicate that our Universe is very close to flat, with $|k| < 0.01$. The metric is then given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a(t)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & -a(t)^2 r^2 \sin^2\theta \end{pmatrix}.$$

The scale factor is dimensionless and is often normalized such that $a(t_0) = 1$ today at time t_0.

By the way, the inverse metric $g^{\mu\nu}$ is defined such that $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$, where δ^μ_ν is the Kronecker delta. It is given by

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1-kr^2}{a(t)^2} & 0 & 0 \\ 0 & 0 & \frac{-1}{a(t)^2 r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{a(t)^2 r^2 \sin^2\theta} \end{pmatrix}.$$