1 Neutrino oscillations beyond plane waves

We have seen that neutrino oscillations arise when a superposition of neutrino mass eigenstates is produced, propagates macroscopic distances, and scatters through a charged-current process to produce a charged lepton that is not necessarily of the same flavor as the one at production. Given sufficiently long baselines, neutrino flavor transitions may always occurs due to mixing, but if we are to see oscillatory phenomena, namely one where the probabilities can rise and fall as a function of the baseline and time, then coherence must be preserved throughout the process. This is the hallmark of a quantum mechanical interference effect and is the reason why we can refer to neutrino oscillation experiments as Earth-sized flavor interferetometers. Otherwise, if we were always to see a constant transition probability, we would not be able to claim that neutrinos *oscillate*, but rather that they simply mix.

In this section, we will make these statements more precise and re-derive the standard formula for the oscillation probability $P(\nu_\alpha \to \nu_\beta)$ in vacuum, now accounting for the localization of the neutrinos. It should feel jarring that we have derived the oscillation formula assuming that neutrinos are described by plane waves. Plane waves are fully delocalized in space and time, and therefore have a perfectly defined momentum and energy. So how can we talk about neutrinos being produced at a source and detected at a detector, separated by a macroscopic distance L? Here, we will localize the neutrino with a wave packet description, which will bring with it two major conceptual improvements: i) we will not need to assume neutrino mass eigenstates have the same energy or momentum, and ii) the conditions for coherence of the mass eigenstates to be maintained will naturally arise from our assumptions about the wavepacket size.

More sophisticated treatments in QFT, often called the *external* wave packet approach, have been long known in the literature [1–3], and their results have been shown to be directly mapped onto the *internal* wave packet approach [4] we discuss here. Nonetheless, neutrino oscillations are notorious for being conceptually confusing and you can still find many debates on arXiv. I point you to Ref. [5], for example, for a nice modern discussion of the oscillation phenomenon. We can seek consolation in the fact that many aspects are common to all approaches. For instance, all approaches admit the the need for momentum uncertainties in the initial and final neutrino processes and, sooner or later, all calculations rely on the ultra-relativistic nature of the mass states through expansions of $\sqrt{m^2+p^2} \simeq p + \frac{m^2}{2p}$.

A more accurate picture of flavor oscillations. We start by reframing the oscillation problem. While before we phrased it as a transition between flavor states, one can avoid the use of flavor states altogether and focus on the overall chain of events in an oscillation measurement. The setup we have in mind is shown in Fig. 1: a subprocess $P_{\rm I} \to \ell_\alpha \nu_i X_{\rm I}$ happens at the source and a subprocess $\nu_i P_{\rm F} \to \ell_\beta X_{\rm F}$ happens at the detector, both mediated by a long-distance propagator of a neutrino mass eigenstate ν_i. Overall, there are really only two initial states: the production particle $P_{\rm I}$ and the detection particle $P_{\rm F}$. There are many final states, however, and these include the charged leptons ℓ_α and ℓ_β,

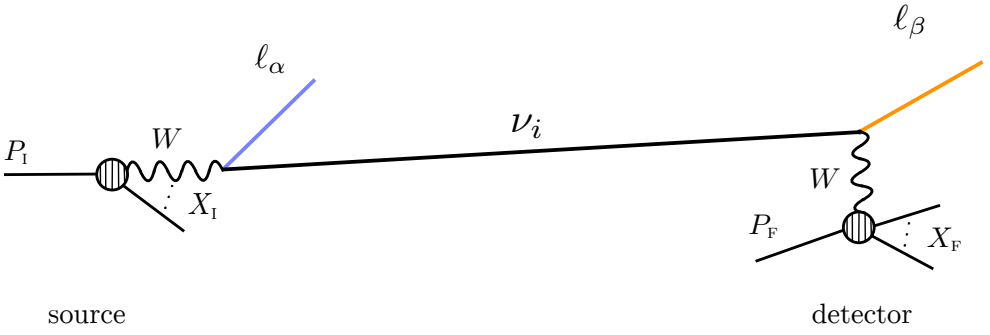

Figure 1: The usual set-up of an oscillation experiment. We show the source, where a process $P_{\rm I} \to \nu_\alpha X_{\rm I}$ happens, where $P_{\rm I}$ and $X_{\rm I}$ may be any particle. For example, for pion decay $\pi^+ \to \nu_i \ell_\alpha^+$, $P_{\rm I} = \pi^+$ and $X_{\rm I}$ is not needed. At the detector location, we show the scattering process $\nu_i P_i \to \ell_\beta X_{\rm f}$. What is important is that the initial and final states at both the source and detector are flavor-specific, as they involve charged leptons of definite flavor ℓ_α and ℓ_β.

as well as other hadronic or leptonic states that may serve purely as spectators. Overall, what we care about is the amplitude for the full process

$$P_{\rm I}(x_1) + P_{\rm F}(x_2) \to \ell_\alpha(x_1) + \ell_\beta(x_2) + \dots \tag{1.1}$$

where the two charged leptons are associated with different space-time coordinates $x_1 = (t_1, \vec{x_1})$ and $x_2 = (t_2, \vec{x_2})$. The dots indicate other hadronic or leptonic states that may or may not be detected.

For instance, in a classic pion decay experiment, we have $P_{\rm I} = \pi^+$ and $X_{\rm I}$ is not needed, while at the detector, $P_{\rm F}$ may be a neutron and $X_{\rm F}$ may be a final-state proton. The sub-processes are then $\pi^+ \to \mu^+ \nu_\mu$ at the source and $\nu_e n \to e^- p$ at the detector. Using the neutrinos as an intermediate state, the full process is

$$\pi^+(\text{source}) + \text{n}(\text{detector}) \to \mu^+(\text{source}) + \text{e}^-(\text{detector}) + \text{p}^+(\text{detector}). \tag{1.2}$$

In summary, what oscillation experiments are measuring are two separate CC interactions, one at the source and one at the detector, separated by a macroscopic distance $\vec{L} = \vec{x_2} - \vec{x_1}$, over which we expect intermediate neutrino states to propagate for a time $T = t_2 - t_1$. These neutrinos are just an intermediate line in a Feynman diagram, but it is important that they propagate a macroscopic distance. Note that this is not usually how we think about Feynman diagrams, as they usually describe microscopic processes. In this case, however, the neutrino line is special as it is "on-shell" and it will give a sizeable effect as long as $L - v_i T \lesssim \sigma_x$, where σ_x is the spatial size of the neutrino wave packet and v_i is the group velocity of the mass eigenstate ν_i. This is the condition that the neutrino wave packet overlaps with the detection process. We will come back to this important point later.

In this language, the oscillation phenomenon is the observation that the amplitude for two charged-current (CC) interactions separated in space shows an oscillatory dependence on the distance $\vec{L}$ and time T between them.

We may also be tempted to draw an analogy with the double-slit experiment: the flavor of the neutrinos is like the intensity of light on a screen, and the phase difference between neutrino mass eigenstates is like the phase difference acquired by photons traveling through different paths in the two slits. However, unlike light intensity, neutrino flavor is divided into three discrete states, the proportions of each varying continuously across different locations. In this sense, this is a richer phenomenon than the double-slit experiment, as we have three types of waves that can interfere with each other to produce a non-trivial flavor pattern.Neutrino coherent forward scattering and its index of refraction

Because of the quantum mechanical uncertainty in the production and detection space-time coordinates, we may also *not* assume that the neutrino mass eigenstates propagate exactly the same distance and time, but rather that these propagated for a typical time $T \pm \delta t$ and distance $\vec{L} \pm \delta\vec{x}$. As long as δt and δx are small enough, the amplitudes for different neutrino mass eigenstates add up coherently and oscillations may be observed.

A similar statement holds for the energy and momentum of each mass eigenstate: these may also vary within some quantum-mechanical uncertainty δE and δp, but, as long as these variations are small enough, coherence is preserved and oscillations may be observed.

A corollary of this is that if your source or detection processes were able to somehow differentiate between the different mass eigenstates, then coherence would be lost and oscillations would be lost (we will prove this below). One trivial example of this is if only one mass eigenstate were kinematically allowed to be produced in pion decay (say $m_{\nu_2}, m_{\nu_3} > m_\pi - m_\mu$). In that case, no oscillations would be observed because only one mass eigenstate is produced and detected. In that sense, we lucked out that such macroscopical quantum-mechanical effects can be observed in nature.

Wave packets in the oscillation formula. We will first discuss the role of production and detection processes, and then later see how the oscillations arise. Initially, the decay of P_I produces some state $|\nu_\alpha\rangle^P$ at the source, where

$$|\nu_\alpha\rangle^P \;=\; N_P \sum_{k,h} \int \frac{d^3p}{(2\pi)^3}\, \mathcal{A}^P_{\alpha k}(\vec{p}, h)\; |\nu_k(\vec{p}, h)\rangle\,, \tag{1.3}$$

where N_P is a factor that normalizes the state, k is a mass index, h is a helicity index, and $\mathcal{A}^P_{\alpha k}(\vec{p}, h)$ is the amplitude for producing $\nu_k^{h=\pm\frac{1}{2}}$ with momentum $\vec{p}$ alongside the charged lepton ℓ_α. The momentum integral accounts for all possible momenta the neutrino may

have, as the production process does not necessarily produce a monochromatic neutrino beam. Energy and momentum are related as usual, $E_k = \sqrt{|\vec{p}|^2 + m_k^2}$. The amplitude may be computed in pertubation theory,

$$\mathcal{A}^P_{\alpha k}(\vec{p}, h) = \langle \nu_k(\vec{p}, h), \ell_\alpha, X_{\rm I}| \, \hat{\mathcal{H}}_{\rm int} \, |P_{\rm I}\rangle \,, \tag{1.4}$$

describing the transition $P_{\rm I} \to \nu_k^{h=\pm\frac{1}{2}} + \ell_\alpha + X_{\rm I}$.

Note that the definition above is very different from the usual flavor state $|\nu^\alpha\rangle = U^*_{\alpha i} |\nu_i\rangle$. We can make a connection between the two by noting that

$$A_{\alpha k}(\vec{p}, h) = U^*_{\alpha k} M_{\alpha k}(\vec{p}, h), \tag{1.5}$$

where we just factored out the mixing matrix element $U^*_{\alpha k}$ factor explicitly. The matrix element $M_{\alpha k}(\vec{p}, h)$ contains all the kinematic and dynamical information about the production process. If all mass eigenstates have very similar mass and the production process is not sensitive to these differences, then we may approximate $M_{\alpha k}(\vec{p}, h) \approx M_\alpha(\vec{p}, h)$.

In that case, we can factor out $M_\alpha(\vec{p}, h)$ from the sum over k in Eq. (1.3) and write

$$|\nu_\alpha\rangle^P = N_P \sum_h \int \frac{d^3p}{(2\pi)^3} \, M_\alpha(\vec{p}, h) \, \left(\sum_k U^*_{\alpha k} \, |\nu_k(\vec{p}, h)\rangle \right). \tag{1.6}$$

and by virtue of the unitarity of the mixing matrix and the ultra-relativistic nature of neutrinos, the normalization factor

$$N_P^{-2} = \sum_h \int \frac{d^3p}{(2\pi)^3} |M_\alpha(\vec{p}, h)|^2 \left(\sum_k |U_{\alpha k}|^2 \right) \simeq \int \frac{d^3p}{(2\pi)^3} |M_\alpha(\vec{p})|^2. \tag{1.7}$$

where we neglected flipped-helicity terms suppressed by m_k^2/E_k^2 in the last step.

Now, one is left to compute the functions $M_{\alpha k} \sim M_\alpha$. This could be a rather involved process, but we should note a lot of the complication in $M_{\alpha k}$ is going to cancel with the normalization factor N_P. In fact, all prefactors in the amplitude (including G_F!) will cancel out, leaving us only with the dependence on the neutrino momentum. This momentum dependence can, however, also be quite complicated. Recall that we are interested in a neutrino state with a localized wavefunction, which means it came from a localized production process (the decay of a localized pion, for instance), so to faithfully reproduce it would require us to know something about the localization and dynamics of the production process, all of which is usually modelled by wave-packets as well. While calculating the exact form of $M_\alpha(\vec{p})$ is quite involved, it is usually assumed that the neutrino also has a given wave packet shape with some unknown width, which we can argue about phenomenologically. That is, whatever the production process is, we can assume it produces neutrinos with a wave packet shape $\psi^P_\alpha(\vec{p})$ centered around some momentum $\vec{p}_\alpha$ with some spread σ^P_p (see, e.g., Ref. [4]).

An analogous discussion holds for the detection process $\nu_\beta \, P_{\rm F} \to \ell_\beta \, X_{\rm F}$, where a detection flavor state $|\nu_\alpha\rangle^D$ with an amplitude for detection $A^D_{\alpha k}(\vec{p}, h)$ can be defined. Note that this is conceptually a different state from the time-evolved production state $|\nu_\alpha(t)\rangle^P$.

The coefficients $A^D_{\alpha k}(\vec{p}, h)$ are computed through the amplitude for the detection process, which in our formalism, is completely ignorant about the existence of the production state $|\nu_\alpha(t)\rangle^P$. Therefore, in detection, we can assume as well that the detection flavor state is given by a wave packet $\psi^D_\beta(\vec{p})$ centered around some momentum $\vec{p}_\beta$ with some spread σ^D_p, all of which dictated by the Weak interactions of the neutrino with localized nuclei in the detector. With these assumptions, we can finally move on to calculate some flavor-change amplitude.

Note the normalization of the flavor state is a clear sign that we are working in a quantum mechanical description. To compute probabilities, we rely on normalized states. In a QFT description, however, the normalization is not necessary, but neither is the concept of $P_\mathrm{I} \to X_\mathrm{I}$ in the first place. There, the full process $P_\mathrm{I}\, P_\mathrm{F} \to X_\mathrm{I}\, X_\mathrm{F}\, \ell_\alpha\, \ell_\beta$ in Fig. 1 can be compute directly through the use of long-distance propagators. If the production, propagation and detection parts of the amplitude squared factorize, an object analogous to the oscillation probability can be extracted. This factorization is implicitly assumed in our calculation.

Note that the decay rate of the P_I particle can be computed as

$$\left|\mathcal{A}^P\right|^2 = \left|\langle \nu_\alpha(\vec{p}, h), \ell_\beta, X_\mathrm{I}|\, \hat{\mathcal{H}}_\mathrm{int}\, |P_\mathrm{I}\rangle\right|^2 = \sum_{k,h} |U_{\alpha k}|^2 \int \mathrm{d}^3 p\, \left|M^P_{\alpha k}(\vec{p}, h)\right|^2, \tag{1.8}$$

and it becomes evident that the decay rate is given by the *incoherent* sum of the decay rate into different massive neutrinos. No interference is present as the states $|\nu_k\rangle$ are assumed to be orthonormal to each other. This remains true in the QFT description [3].

Wavepackets. To summarize, in our calculation we will assume that *i)* flipped-helicity terms ($h = +1$ for neutrinos), suppressed by m_k^2/E_k^2, are negligible, *ii)* all neutrinos travel in the same direction, $\vec{p} \to p$, and *iii)* the production and detection processes are not sensitive to the neutrino mass differences, amounting to replacing $M_{\alpha k} \approx M_\alpha$. Under these assumptions, we are justified to take normalized gaussian wave packets for production and detection flavor states as the *ansatz*,

$$|\nu_\alpha\rangle^i = \sum_k U^*_{\alpha k} \int \mathrm{d}p\, \psi^i_k(p)\, |\nu_k(p)\rangle\,, \quad \psi^i_k(p) = \left(2\pi\, \sigma_p^{i\,2}\right)^{-1/4} \exp\left[-\frac{(p-p_k)^2}{4\sigma_p^{i\,2}}\right], \tag{1.9}$$

with σ^i_p being the spread around the central momenta p_k and $i = P, D$.

The flavor-change amplitude for baseline L and time T is then calculated evolving the state both in space and time (finally recovering a covariant description!),

$$\mathcal{A}_{\alpha\beta}(L, T) = \left\langle \nu^D_\beta \right|\, e^{-i\hat{E}T + i\hat{P}L}\, \left|\nu^P_\alpha\right\rangle \;=\; \mathcal{N} \sum_k U^*_{\alpha k} U_{\beta k}\, I_k(L, T), \tag{1.10}$$

where we colleceted normalization factors into $\mathcal{N}$ and defined the Gaussian integral for a given mass eigenstate k,

$$I_k(L,T) = \int_{-\infty}^{\infty} dp\ \exp\Big[-\frac{(p-p_k)^2}{4\sigma_p^2}\Big]\ \exp\Big[i\,pL - i\,E_k(p)\,T\Big], \tag{1.11}$$

where we defined the combined momentum spread

$$\frac{1}{\sigma_p^2} = \frac{1}{\sigma_p^{P\,2}} + \frac{1}{\sigma_p^{D\,2}}. \tag{1.12}$$

The energy $E_k(p) = \sqrt{p^2 + m_k^2}$ depends on the mass of the eigenstate k and on the multiple momenta p. Expanding this energy around the central momentum p_k will be greatly beneficial to perform the integral,

$$E_k(p)\ \approx\ E_k + v_k(p-p_k), \qquad v_k \equiv \frac{\partial E_k}{\partial p}\Big|_{p_k} = \frac{p_k}{E_k}, \tag{1.13}$$

which is good to $\mathcal{O}\big((\sigma_p/E)^2\big)$.

Continue this derivation with the third-order term to show that the wavepacket will disperse over time!

Rearranging the integral slightly, we get a more suggestive form

$$I_k = e^{\,ip_kL - iE_kT}\int dp\ \exp\left[-\frac{(p-p_k)^2}{4\sigma_p^2} + i(p-p_k)(L-v_kT)\right].$$

For real and positive constants, the Gaussian integral is just

$$\int_{-\infty}^{\infty} dp\ e^{-a(p-p_0)^2+ib(p-p_0)} = \sqrt{\frac{\pi}{a}}\ \exp\left[-\frac{b^2}{4a}\right],$$

with $a = \frac{1}{4\sigma_p^2}$, $b = L - v_kT$. Using this, gives

$$I_k(L,T) = \sqrt{4\pi}\,\sigma_p\ \exp\Big[ip_kL - iE_kT\Big]\ \exp\left[-\frac{(L-v_kT)^2}{4\sigma_x^2}\right], \qquad \sigma_x = \frac{1}{2\sigma_p}. \tag{1.14}$$

Finally, we can write the space-time-dependent flavor-change amplitude as

$$\boxed{\mathcal{A}_{\alpha\beta}(L,T) = \tilde{\mathcal{N}}\sum_k U_{\alpha k}^* U_{\beta k}\ \exp\Big[ip_kL - iE_kT\Big]\ \exp\left[-\frac{(L-v_kT)^2}{4\sigma_x^2}\right].} \tag{1.15}$$

where we absorbed the $\sqrt{4\pi}\,\sigma_p$ factors into the normalization $\tilde{\mathcal{N}}$. This is the wave-packet generalization of the oscillation amplitude. Note that each mass eigenstate k comes with a complex phase $\exp[ip_kL - iE_kT]$ which is then potentially damped by the real exponential $\exp\big[-(L-v_kT)^2/(4\sigma_x^2)\big]$ that encodes the localization of the neutrino wave packet. This real exponential factor is crucial, as it ensures that the amplitude is only sizeable when

the neutrino wave packet overlaps with the production and detection processes, i.e., when $L \approx v_k T$ within a spread σ_x. Neutrinos oscillate between a production and detection *regions*, not points!

The amplitude depends on time. This is not surprising: for the same distance L, we expect the oscillation amplitude to modulate as we dial the time T back and forth, thereby allowing the wave to propagate.

So, what now? Our original formulae made use of $L \simeq T$, for $v_k \approx 1$. This was a major simplification to say the least, but in what follows we will see how to recover the usual oscillation formula without making this assumption under the condition of ultra-relativistic neutrinos. To make contact with the usual formalism, we recall that experiments do not measure time T directly, but rather the distance L between source and detector. Therefore, T is an unmeasured variable, and we must integrate (average) over it to obtain the event rate as a function of L only. Note that the real exponentials will ensure that the integral over T is finite and that the integrand will only be sizeable when T is close to L/v_k to within σ_x.

The oscillation probability. Integrating the square of the amplitude over T, we find the oscillation probability as a function of L,

$$P(\nu_\alpha \to \nu_\beta; L) = \int_{-\infty}^{\infty} dT \, \left|\mathcal{A}_{\alpha\beta}(L,T)\right|^2. \tag{1.16}$$

Using Eq. (1.15), we have

$$P(\nu_\alpha \to \nu_\beta; L) = |\tilde{\mathcal{N}}|^2 \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} \, \mathcal{I}_{kj}(L), \tag{1.17}$$

where we defined the time integral for the interference term between mass eigenstates k and j,

$$\mathcal{I}_{kj}(L) = \int_{-\infty}^{\infty} dT \, \exp\left[-\frac{(L-v_k T)^2 + (L-v_j T)^2}{4\sigma_x^2}\right] \exp[-i\,(\Delta E_{kj})T] \exp[i\,(\Delta p_{kj})L] \tag{1.18}$$

with $\Delta E_{kj} = E_k - E_j$ and $\Delta p_{kj} = p_k - p_j$. The last exponential is just a phase factor that can be factored out of the integral. The integral over T is Gaussian and can be done analytically. We write the real exponent as a quadratic form

$$\frac{(L-v_k T)^2 + (L-v_j T)^2}{4\sigma_x^2} = AT^2 + BT + C, \quad A = \frac{v_k^2 + v_j^2}{4\sigma_x^2}, \; B = -\frac{L(v_k+v_j)}{2\sigma_x^2}, \; C = \frac{L^2}{2\sigma_x^2}.$$

Complete the square and use

$$\int dT \, e^{-A(T-T_*)^2} \, e^{-i\Delta E\, T} = \sqrt{\frac{\pi}{A}} \, \exp\left[-\frac{(\Delta E)^2}{4A}\right] \, e^{-i\Delta E\, T_*},$$

where we defined the stationary time

$$T_* = -\frac{B}{2A} = \frac{L(v_k+v_j)}{(v_k^2+v_j^2)}. \tag{1.19}$$

Note that for $v_k \approx v_j \approx v$, we have $T_* \approx L/v$, which is not surprising. We will not make this approximation yet, however. Putting everything together, including the term $e^{-(C-B^2/(4A))}$ from completing the square, we find

$$\mathcal{I}_{kj}(L) = \sqrt{\frac{4\pi\sigma_x^2}{v_k^2+v_j^2}}\ \exp\left[-\frac{(\Delta E_{kj})^2\,\sigma_x^2}{v_k^2+v_j^2}\right]\ \exp\left[-\frac{(v_k-v_j)^2\,L^2}{4\sigma_x^2\,(v_k^2+v_j^2)}\right]\ e^{-i\Delta E_{kj}T_*}e^{i\Delta p_{kj}L}, \quad (1.20)$$

where we can already spot the single complex phase that matters for oscillations, as well as two real exponentials that will damp the oscillatory behavior.

The oscillation phase is given by

$$\Delta\phi = \Delta p_{kj}L - \Delta E_{kj}T_*.$$

In the ultrarelativistic regime, we can write the energy and momentum of each mass eigenstate as

$$E_k \simeq E + \xi\frac{m_k^2}{2E}, \qquad p_k \simeq E - (1-\xi)\frac{m_k^2}{2E}, \qquad L_{kj}^{\rm osc} \equiv \frac{4\pi E}{|\Delta m_{kj}^2|},$$

for some $0 \leq \xi \leq 1$ that encodes how the production/detection process correlates E and p. Note that this is quite a generic parametrization: $\xi = 0$ corresponds to the equal-momentum assumption, $\xi = 1$ to the equal-energy assumption, and $\xi = 1/2$ to the average of the two. Using these expansions, we can write

$$\Delta E_{kj} \simeq \xi\frac{\Delta m_{kj}^2}{2E}, \qquad \Delta p_{kj} \simeq -(1-\xi)\frac{\Delta m_{kj}^2}{2E},$$

such that

$$\Delta\phi = \frac{\Delta m_{kj}^2 L}{2E}\left(1 - \frac{\xi(v_k+v_j)}{v_k^2+v_j^2}\right). \quad (1.21)$$

In the ultrarelativistic limit, we also have

$$v_k \simeq 1 - \frac{m_k^2}{2E^2}, \qquad v_k - v_j \simeq -\frac{\Delta m_{kj}^2}{2E^2},$$

so $v_k^2 + v_j^2 \simeq 2$, $v_k + v_j \simeq 2$, giving us the so sought-after

$$\boxed{\Delta\phi \simeq \frac{\Delta m_{kj}^2 L}{2E} = 2\pi\frac{L}{L_{kj}^{\rm osc}}.} \quad (1.22)$$

That is the standard oscillation phase, independent of ξ at leading order! And this justifies a posteriori the usual assumptions of equal energy or equal momentum, as they lead to the same oscillation phase, all thanks to the ultrarelativistic nature of neutrinos.

Putting it all together, the vacuum oscillation probability with wave-packet effects is

$$\boxed{P(\nu_\alpha \to \nu_\beta) = \sum_{k,j} U_{\alpha k}^* U_{\beta k}\, U_{\alpha j} U_{\beta j}^*\ \exp\left[-2\pi i\frac{L}{L_{kj}^{\rm osc}}\right]\ P_{kj}^{\rm coh}\, P_{kj}^{\rm loc},} \quad (1.23)$$

with

$$P_{kj}^{\rm coh} = \exp\left[-\left(\frac{L}{L_{kj}^{\rm coh}}\right)^2\right], \qquad P_{kj}^{\rm loc} = \exp\left[-2\pi^2\,\xi^2\left(\frac{\sigma_x}{L_{kj}^{\rm osc}}\right)^2\right], \tag{1.24}$$

and

$$\boxed{L_{kj}^{\rm osc} \equiv \frac{4\pi E}{\Delta m_{kj}^2}} \qquad \boxed{L_{kj}^{\rm coh} \equiv \frac{4\sqrt{2}\,E^2\,\sigma_x}{\Delta m_{kj}^2}}, \tag{1.25}$$

where we used $v_k^2 + v_j^2 \simeq 2$ in the ultrarelativistic limit to simplify the expressions. Let us comment on the meaning of each factor.

The oscillation length $L_{kj}^{\rm osc}$ determines the wavelength of the oscillations. The larger the oscillation length, the longer it takes for oscillations to develop.

Wave-packet separation (coherence) factor: The coherence length determines the distance over which wave-packet separation occurs. Beyond this length, different mass eigenstates no longer overlap and oscillations are suppressed. The larger the coherence length, the longer it takes for wave packets to separate and the less suppressed oscillations become (oscillations are observable when $L \ll L_{kj}^{\rm coh}$).

$$P_{kj}^{\rm coh} = \exp\left[-\frac{(v_k - v_j)^2\,L^2}{4\sigma_x^2\,(v_k^2 + v_j^2)}\right] = \exp\left[-\left(\frac{L}{L_{kj}^{\rm coh}}\right)^2\right]. \tag{1.26}$$

Localization factor. This requires that $\sigma_x \ll L_{kj}^{\rm osc}$ so that the wavepacket does not overlap with the entire baseline. It can be shown that this is equivalent to the requirement that production/detection do not resolve the tiny energy differences between m_k and m_j.

$$P_{kj}^{\rm loc} = \exp\left[-\frac{(\Delta E_{kj})^2\,\sigma_x^2}{v_k^2 + v_j^2}\right] \simeq \exp\left[-2\pi^2\xi^2\left(\frac{\sigma_x}{L_{kj}^{\rm osc}}\right)^2\right]. \tag{1.27}$$

Can oscillations be observed if we can distinguish between different mass eigenstates at production or detection? Let us consider a tiny mass difference Δm_{kj}^2 and an uncertainty on a given mass:

$$\delta m_k^2 = \sqrt{(2E_k\delta E_k)^2 + (2p_k\delta p_k)^2} \approx 2\sqrt{2}E\sigma_p, \tag{1.28}$$

If $\delta m_k^2 \gg |\Delta m_{kj}^2|$, then the production/detection processes cannot distinguish between mass eigenstates k and j, and oscillations may be observed. Conversely, if $\delta m_k^2 \ll |\Delta m_{kj}^2|$, then the production/detection processes can distinguish between mass eigenstates k and j, and oscillations are suppressed. The condition $\delta m_k^2 \gg |\Delta m_{kj}^2|$ can be rewritten as

$$\sigma_x \ll \frac{2\sqrt{2}E}{|\Delta m_{kj}^2|} = \frac{L_{kj}^{\rm osc}}{2\pi}, \tag{1.29}$$

which is exactly the localization condition derived above.

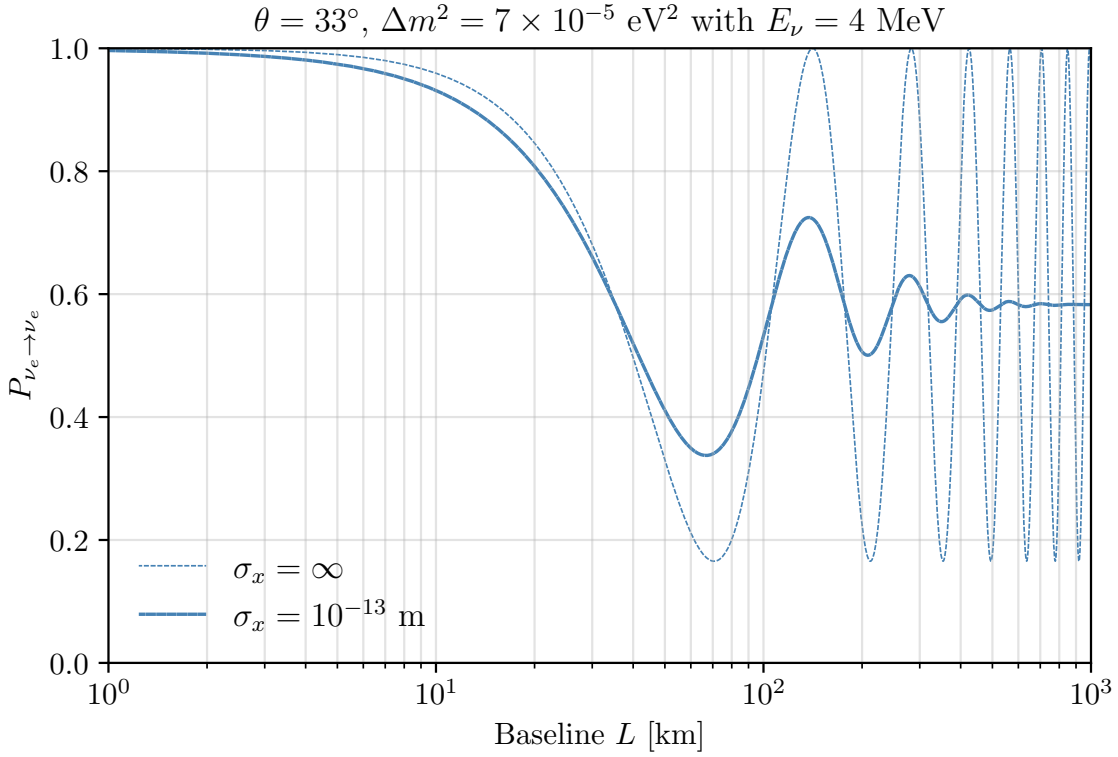

Figure 2: Illustration of wave-packet separation and the damping of oscillations.

A more useful form of the probability shows how the oscillatory behavior is damped by the two real exponentials,

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 2\sum_{k>j} \mathrm{Re}\left(A^{kj}_{\alpha\beta}\right) \left[1 - \cos\left(\frac{\Delta m^2_{kj} L}{2E}\right) P^{\mathrm{coh}}_{kj} P^{\mathrm{loc}}_{kj}\right] \tag{1.30}$$
$$+ 2\sum_{k>j} \mathrm{Im}\left(A^{kj}_{\alpha\beta}\right) \sin\left(\frac{\Delta m^2_{kj} L}{2E}\right) P^{\mathrm{coh}}_{kj} P^{\mathrm{loc}}_{kj}.$$

where we used again $A^{kj}_{\alpha\beta} = U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}$. This is significant! The *oscillation pattern* is damped by two real exponentials that depend on the distance L and on the wave-packet size σ_x. If either of these exponentials is very small, oscillations are suppressed, although flavor change may still occur through the non-oscillatory terms (mixing $\neq$ oscillations). Every time we have observed oscillations, we have had $L \ll L^{\mathrm{coh}}_{kj}$ and $\sigma_x \ll L^{\mathrm{osc}}_{kj}$, so that both real exponential factors are ≈ 1 and one recovers the standard plane-wave formula.

Two-flavor limit (for intuition). We can recover the two-flavor limit to gain some intuition. With mixing angle θ and Δm^2, we cannot put it in the form of two sine functions anymore, but we can write for $\alpha \neq \beta$,

$$P(\nu_\alpha \to \nu_\beta) = \frac{\sin^2(2\theta)}{2} \left[1 - \cos\left(\frac{\Delta m^2 L}{2E}\right) \exp\left[-\left(\frac{L}{L^{\mathrm{coh}}}\right)^2\right] \exp\left[-2\pi^2 \xi^2 \left(\frac{\sigma_x}{L^{\mathrm{osc}}}\right)^2\right]\right],$$

with $L^{\rm coh}, L^{\rm osc}$ as above. This makes the physics of the two damping factors transparent. An illustration of the damping of oscillations due to wave-packet separation is shown in Fig. 2.

References

[1] C.Y. Cardall, *Coherence of neutrino flavor mixing in quantum field theory*, *Phys. Rev.* **D61** (2000) 073006.

[2] M. Beuthe, *Oscillations of neutrinos and mesons in quantum field theory*, *Phys. Rept.* **375** (2003) 105.

[3] C. Giunti, *Neutrino wave packets in quantum field theory*, *JHEP* **11** (2002) 017.

[4] E.K. Akhmedov and J. Kopp, *Neutrino Oscillations: Quantum Mechanics vs. Quantum Field Theory*, *JHEP* **04** (2010) 008.

[5] A. Kobach, A.V. Manohar and J. McGreevy, *Neutrino Oscillation Measurements Computed in Quantum Field Theory*, *Phys. Lett.* **B783** (2018) 59.