1 Neutrino oscillations

Let me start with some definitions. The quantum mechanical state of a neutrino of a given flavor is denoted by:

$$|\nu_\alpha\rangle\,. \tag{1.1}$$

Here the flavor $\alpha = e, \mu$ or τ is determined by the charged lepton with which the neutrino interacts. They are defined by what charged lepton they appear together with in a weak interaction process. However, they do not have a determined mass. When neutrinos propagate, they travel in a superposition of different states called neutrino mass states. They are represented as

$$|\nu_i\rangle\,, \tag{1.2}$$

where $i = 1, 2, 3$ identifies the mass of the neutrino. That is, $|\nu_1\rangle$ has a mass m_1, and $|\nu_2\rangle$ has a mass m_2, and so on. Because they have well-defined masses, these are the eigenstates of the Hamiltonian at least in vacuum, where their energy is simply $E_i = \sqrt{p^2 + m_i^2}$. If neutrinos were interacting with some external potential, then the mass states are no longer necessarily the eigenstates of the Hamiltonian, as we will see.

The correct way to think about it is to think of flavor states as made-up mathematical objects and consider only the mass states as physical particles. In this picture, when pions decay for example, they are actually producing all neutrino mass states ($\pi^+ \to \mu^+\nu_1$, $\pi^+ \to \mu^+\nu_2$, and $\pi^+ \to \mu^+\nu_3$), but some mass states are produced more often than others, depending on their affinity with the charged-lepton companion. For instance, the first neutrino, ν_1, has a higher affinity with muons than ν_2, and so the object $|\nu_\mu\rangle$ contains more $|\nu_1\rangle$ than $|\nu_2\rangle$. Conversely, we also say that ν_1 has a larger ν_μ component than ν_2. Because neutrino masses are so small and close together, the uncertainty on the energy and momentum of the final states covers that entire difference, meaning the neutrino mass states are produced in a coherent superposition. Then, the different neutrino mass propagate almost in synchrony, but with slight differences in their phases that accumulate over time. This is the entire premise of the oscillation phenomenon.

The question is the following: if I produce a neutrino of a given flavor, say ν_μ, and let it propagate for some time t, what is the probability that I will detect it as a different flavor, say ν_e? This should be straightforward to answer with the probability:

$$P(\nu_\mu \to \nu_e) = |A_{\nu_\mu \to \nu_e}|^2 = |\,\langle \nu_e | \nu_\mu(t)\rangle\,|^2, \tag{1.3}$$

Let us solve this in a very simple setup first.

Two-flavor oscillations. Assume that there are only two neutrinos. I can count them as flavor states or as mass states: ν_e and ν_μ, or ν_1 and ν_2. When I produce a flavor state in my experiment, I produce a superposition of states:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle\,, \tag{1.4}$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle\,, \tag{1.5}$$

where the θ is some angle that controls how misaligned the neutrino mass and flavor states are. In this convention, $\theta = 0$ means that the ν_e is just the first mass state. Both the flavor and the mass bases form an orthonormal basis:

$$\langle \nu_e | \nu_e \rangle = 1, \quad \langle \nu_\mu | \nu_\mu \rangle = 1, \text{ and } \langle \nu_e | \nu_\mu \rangle = 0, \tag{1.6}$$

$$\langle \nu_1 | \nu_1 \rangle = 1, \quad \langle \nu_2 | \nu_2 \rangle = 1, \text{ and } \langle \nu_1 | \nu_2 \rangle = 0. \tag{1.7}$$

Also, note that we only need one parameter to describe this mixing. Had we written it more generally as $|\nu_e\rangle = c_1 e^{i\phi_1} |\nu_1\rangle + c_2 e^{i\phi_2} |\nu_2\rangle$ and $|\nu_\mu\rangle = c_3 e^{i\phi_3} |\nu_1\rangle + c_4 e^{i\phi_4} |\nu_2\rangle$, with real c_i and ϕ_i, then the orthonormality conditions would have forced

$$\langle \nu_e | \nu_e \rangle = c_1^2 + c_2^2 = 1, \quad \langle \nu_\mu | \nu_e \rangle = c_3 c_1 e^{i\Delta\phi_{31}} + c_4 c_2 e^{i\Delta\phi_{42}} = 0,$$
$$\langle \nu_\mu | \nu_\mu \rangle = c_3^2 + c_4^2 = 1, \quad \langle \nu_e | \nu_\mu \rangle = c_1 c_3 e^{i\Delta\phi_{13}} + c_2 c_4 e^{i\Delta\phi_{24}} = 0,$$

where $\Delta\phi_{ij} = \phi_i - \phi_j$. We may also argue that the phase of an individual mass state $|\nu_i\rangle$ can be absorbed into the definition of the state itself, so we are justified in absorbing at least two phases (you can only do this once and in one specific basis). In this case, we started with eight parameters, found four constraints, and then eliminated two phases by redefining the states, reducing the problem to simply one real parameter and one complex phase. There are many ways to parameterize this system, but one particularly convenient one is through a rotation matrix and one phase matrix. We will pick the complex phases such that ν_2 always appears with a phase $e^{i\eta_2} |\nu_2\rangle$:

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\eta_2} \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{pmatrix}. \tag{1.8}$$

This is the most general parameterization for the 2 neutrino scenario. Later, we will show that the phase η_2 does not affect oscillations and, in fact, is only well-defined for Majorana neutrinos, hence it is called a *Majorana phase*.

After some time t has passed, a neutrino mass state *evolves* in time. We denote the time-evolved state as $|\nu_i(t)\rangle$. Because the mass eigenstates are energy eigenstates of the Hamiltonian (in vacuum), the time evolution is quite simply

$$|\nu_1(t)\rangle = e^{-i\hat{H}t} |\nu_1\rangle = e^{-iE_{\nu_1}t} |\nu_1\rangle, \tag{1.9}$$

and similarly for ν_2. Since $e^{-iE_{\nu_1}t}$ is just a number, we note that

$$\langle \nu_i | \nu_j(t) \rangle = \langle \nu_i | e^{-iE_{\nu_j}t} |\nu_j\rangle = e^{-iE_{\nu_j}t} \langle \nu_i | \nu_j \rangle = \delta_{ij} e^{-iE_{\nu_j}t}, \tag{1.10}$$

so the bra-ket is still zero for different mass states. This is not surprising

An energy eigenstate of the Hamiltonian remains an energy eigenstate of the Hamiltonian as it evolves in time. It just picks up a phase. The time-evolved flavor states, however, are *not* energy eigenstates, so $\langle \nu_\alpha | \nu_\beta(t) \rangle$ is not necessarily zero if $\alpha \neq \beta$.

At this point, we have all the ingredients to calculate the oscillation probability

$$P(\nu_\mu \to \nu_e) = |A_{\nu_\mu \to \nu_e}|^2 = |\langle \nu_e | \nu_\mu(t) \rangle|^2. \tag{1.11}$$

Using the properties above, we will calculate the oscillation probability for a muon-neutrino to appear as an electron-neutrino after some time t. The amplitude for the transition is simply,

$$A_{\nu_\mu \to \nu_e} = \langle \nu_e | \nu_\mu(t) \rangle = (\langle \nu_1 | \cos\theta + \langle \nu_2 | \sin\theta) \times (-\sin\theta\, |\nu_1(t)\rangle + \cos\theta\, |\nu_2(t)\rangle)\,. \tag{1.12}$$

Clearly, for $t = 0$, this should be zero. But for $t \neq 0$, we will need to evolve the mass states. Keeping only the non-zero bra-kets, we get

$$\begin{aligned} A_{\nu_\mu \to \nu_e} &= -\cos\theta \sin\theta \langle \nu_1 | \nu_1(t) \rangle + \cos\theta \sin\theta \langle \nu_2 | \nu_2(t) \rangle & (1.13)\\ &= -\cos\theta \sin\theta e^{-iE_{\nu_1} t} \langle \nu_1 | \nu_1 \rangle + \cos\theta \sin\theta e^{-iE_{\nu_2} t} \langle \nu_2 | \nu_2 \rangle & (1.14)\\ &= \cos\theta \sin\theta (e^{-iE_{\nu_2} t} - e^{-iE_{\nu_1} t}). & (1.15) \end{aligned}$$

Factoring out a common phase from the amplitude and using $\sin 2\theta = 2 \sin\theta \cos\theta$, we get

$$A_{\nu_\mu \to \nu_e} = \frac{\sin 2\theta}{2} e^{-iE_{\nu_2} t} (1 - e^{i(E_{\nu_2} - E_{\nu_1})t}) = \frac{\sin 2\theta}{2} e^{-iE_{\nu_2} t} (1 - \cos\Delta_{21} - i \sin\Delta_{21}), \tag{1.16}$$

where we defined the dimensionless phase difference

$$\Delta_{ij} \equiv (E_i - E_j)t. \tag{1.17}$$

The first complex phase will square to one, but the term in the brakets survives:

$$\begin{aligned} P(\nu_\mu \to \nu_e) = |A_{\nu_\mu \to \nu_e}|^2 &= \frac{\sin^2 2\theta}{4} |(1 - \cos\Delta_{21} - i \sin\Delta_{21})|^2 & (1.18)\\ &= \frac{\sin^2 2\theta}{4} \left(1 + 2\cos\Delta_{21} + \cos^2\Delta_{21} + \sin^2\Delta_{21}\right) \\ &= \sin^2 2\theta \sin^2 \frac{\Delta_{21}}{2}. \end{aligned}$$

This is pretty much it. The oscillation probability depends on the mixing angle θ and the phase difference Δ_{21}, which in turn depends on the energy difference between the two mass states and the time of propagation.

By using Eqs. (1.8) and (1.13) and the fact that $|\nu_2\rangle^\dagger = \langle \nu_2|$, show that the Majorana phase η_2 does not appear in the final *amplitude* and, therefore, does not affect the oscillation probability.

We usually work with ultra-relativistic neutrinos, where $E_\nu \gg m_\nu$, so you will typically see this formula in a slightly different form. To get there, we will start by making the naive assumption that all mass state neutrinos have roughly the same momentum $p_1 \simeq p_2 \equiv p$ and expand the energy of each ultra-relativistic eigenstate as

$$E_i = \sqrt{p^2 + m_i^2} \simeq p \left(1 + \frac{m_i^2}{2p^2}\right), \tag{1.19}$$

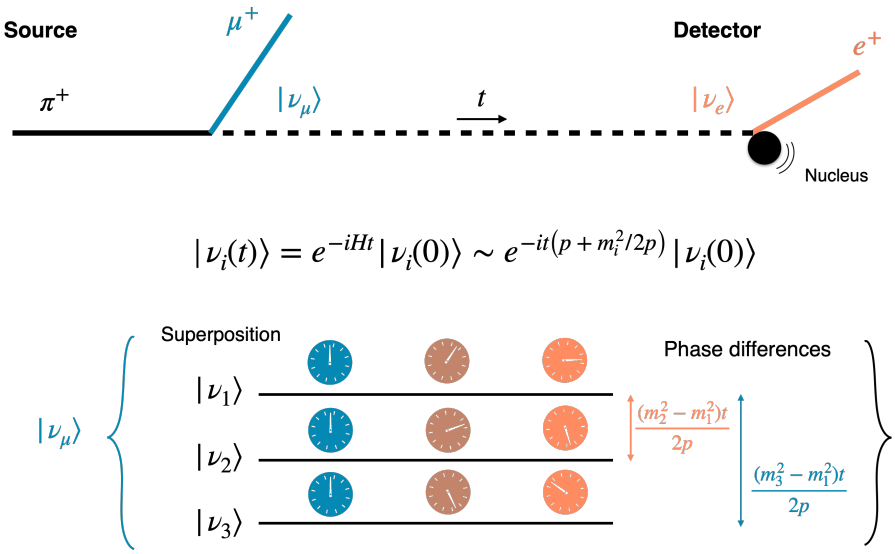


Figure 1: A pictorial representation of the time evolution of a neutrino state produced as a ν_μ. The neutrino is produced as a superposition of mass states ν_1 and ν_2, which evolve with different phases due to their different masses. As time passes, the superposition changes, and the neutrino can be detected as a different flavor, such as ν_e.

which allows us to write

$$\Delta_{ij} \simeq \frac{\Delta m_{ij}^2 t}{2p}, \qquad \text{with} \qquad \Delta m_{ij}^2 \equiv m_i^2 - m_j^2. \tag{1.20}$$

Usually we measure the distance traveled by neutrinos $L = t \times \beta_\nu \simeq t$, where for ultra-relativistic particles $\beta_\nu \simeq 1$. From that, we put the probability in the most familiar form

$$\boxed{P(\nu_\mu \to \nu_e) = \sin^2 2\theta \, \sin^2 \frac{\Delta m_{21}^2 L}{4E_\nu} = \sin^2 2\theta \, \sin^2 \left(\frac{1.27 \times \Delta m_{21}^2[\text{eV}^2] \, L[\text{km}]}{E_\nu[\text{GeV}]} \right).} \tag{1.21}$$

Firstly, it should be clear that in natural units, the argument of the second sine has to be dimensionless. That means that if Δm^2 is expressed in eV^2, as usual, then L would have to be expressed in $1/\text{eV}$ and E in eV. But with natural units we can rewrite the argument in a more convenient way by using $\hbar c = 1 = 197$ fm.MeV, so that $\text{eV}^{-1} = 197 \times 10^{-9}$ m $= 1.97 \times 10^{-10}$ km. Multiplying and dividing by eV^2,

$$\frac{\Delta m_{21}^2 L}{4E_\nu} = \left(\frac{\frac{\Delta m_{21}^2}{\text{eV}^2} \frac{L}{\text{eV}^{-1}}}{4\frac{E_\nu}{\text{eV}}} \right) = 0.25 \times \left(\frac{\frac{\Delta m_{21}^2}{\text{eV}^2} \frac{L}{1.97\times10^{-10}\ \text{km}}}{\frac{E_\nu}{10^{-9}\ \text{GeV}}} \right) = 1.27 \times \left(\frac{\frac{\Delta m_{21}^2}{\text{eV}^2} \frac{L}{\text{km}}}{\frac{E_\nu}{\text{GeV}}} \right). \tag{1.22}$$

Fig. 3 shows how the oscillation probability has a very characteristic dependence on L/E_ν. The first sine term controls the amplitude of the oscillation, while the second sine term controls the frequency of the oscillation. At early times (short distances) or very high energies, the oscillation probability is very small and grows quadratically with L/E_ν.

At very long distances or very low energies, the oscillation probability averages out to $\langle P \rangle = \frac{1}{2}\sin^2 2\theta$, as the sine term oscillates very fast and we cannot resolve the oscillations anymore.

Another convenient way to write the probability is

$$P(\nu_\mu \to \nu_e) = \sin^2 2\theta \, \sin^2\left(\pi \frac{L}{L_{\text{osc}}}\right), \quad \text{with } L_{\text{osc}} = \frac{4\pi E_\nu}{\Delta m^2_{21}} = 2.47 \frac{E_\nu[\text{GeV}]}{\Delta m^2_{21}[\text{eV}^2]}[\text{ km}]. \tag{1.23}$$

with L_{osc} being the oscillation length, i.e., the distance over which the phase difference between neutrino states completes one full cycle, $\Delta_{ij} = 2\pi$. The corollary here is that the oscillation length increases with neutrino energy. This should not be surprising, as higher-energy neutrinos are more relativistic and so it takes them longer to develop a phase difference between the different mass states.

We have only derived what is called the *appearance* probability $P(\nu_\mu \to \nu_e)$, where the final state is different from the initial one. We can also detect oscillations by the *disappearance* of neutrinos by measuring the probability $P(\nu_\mu \to \nu_\mu)$. This is technically speaking the *survival* probability, as we are measuring the probability that a neutrino remains the same flavor after some time, but the oscillation channel is often referred to as just the disappearance channel. The calculation is very similar to the one we did above, and in the 2-neutirno case it is straightforward to show that

$$P(\nu_\mu \to \nu_\mu) = 1 - P(\nu_\mu \to \nu_e) = 1 - \sin^2 2\theta \, \sin^2\left(\frac{\Delta m^2_{21} L}{4E_\nu}\right). \tag{1.24}$$

Note that at short-distances, P is very close to 1, but it can decrease over time, meaning that we can detect the disappearance of muon-neutrinos as they oscillate into electron-neutrinos.

Neutrinos are massive Because we have seen neutrinos oscillate with at least two different independent frequencies, we know that there are at least two neutrinos mass splittings, and therefore at least two neutrinos must be massive!

Octant problem: The 2-neutrino oscillation probability is also invariant under the transformation $\theta \to \theta + \frac{\pi}{2}$, which, by virtue of $\sin(\theta + \pi/2) = \cos(\theta)$ and $\cos(\theta + \pi/2) = -\sin(\theta)$, is equivalent to swapping the flavor eigenstates $\nu_e \leftrightarrow -\nu_\mu$. This means that in a 2-neutrino oscillation regime (in vacuum), we cannot determine which mass eigenstate contains more of the electron or muon flavor.

Mass ordering problem: The 2-neutrino oscillation probability is invariant under the transformation $\Delta m^2_{21} \to -\Delta m^2_{21}$, which is equivalent to swapping the mass eigenstates $\nu_1 \leftrightarrow \nu_2$. This means that in a 2-neutrino oscillation regime (in vacuum), we cannot determine the absolute mass scale of neutrinos, nor can we determine the ordering of the mass states.

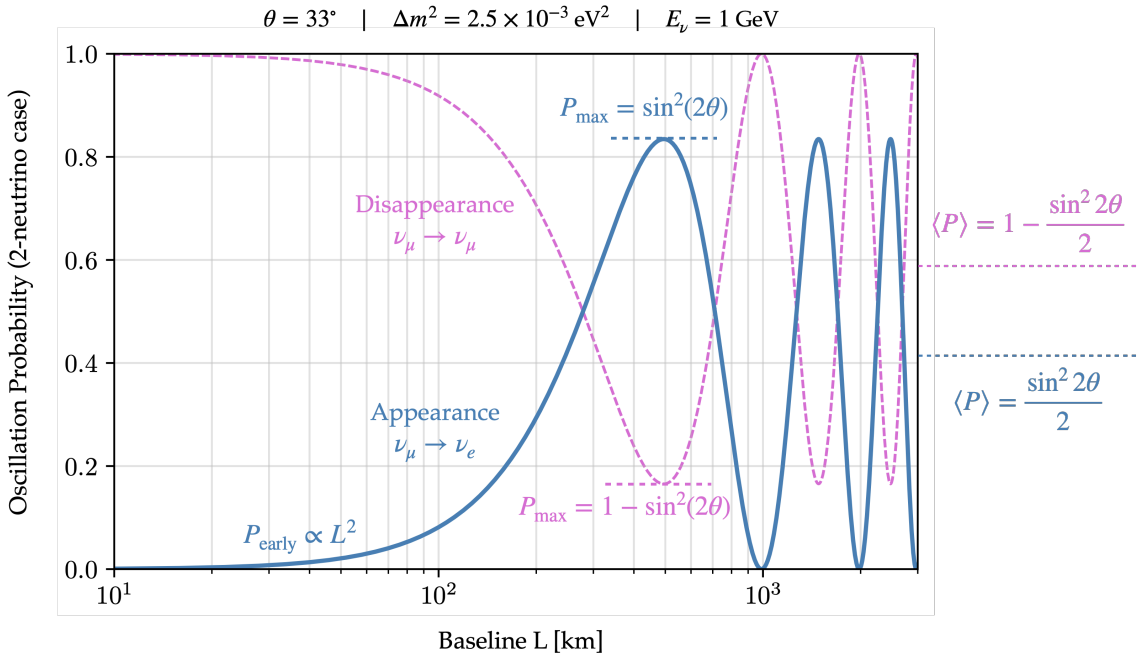


Figure 2: The apperance $P(\nu_\mu \to \nu_e)$ (blue solid) and disappearance $P(\nu_\mu \to \nu_\mu)$ (pink dashed) oscillation probabilities as a function of the baseline L for $E_\nu = 1$ GeV in a 2-neutrino model. We indicate the average probability $\langle P \rangle$ for large baselines.

Repeat the calculation of the oscillation probability assuming that neutrinos have the same *energy* instead, $E \sim p_i + \frac{m_i^2}{2p_i}$, where the momentum of each mass eigenstate is a little different so as to match the difference between their masses. In this case, you will need to keep track of the velocity $v_i = p_i/E$ as well as of the different time of propagation t_i of each mass eigenstate in your calculation, such that $\Delta_{ij} = E_i t_i - E_j t_j$. This assumes the neutrinos propagate the same distance $L = v_i t$, but take different times to get there. Show that you get the same final result for the oscillation probability up to $\mathcal{O}(m^4/E^4)$ corrections.

Now do the same for non-relativistic neutrinos under the assumption of same-momentum but different time of propagation. The form of the oscillation phase $\Delta_{ij} = E_i t_i - E_j t_j$ and show that you do **not** get the same final result due to a factor of 2 difference in the oscillation phase. As we will see, this is not necessarily an inconsitency as the two regimes are physically different, but it demonstrated that the formula we derived is not covariant.

Lorentz invariance of the oscillation probability: As a physical observable, the oscillation probability should be a Lorentz invariant quantity. Let us explore this a little bit more. If we can prove that

$$\Delta_{ij} = E_i t_i - E_j t_j$$

is a Lorentz invariant quantity, then we are done, because the mixing angle θ is just a number. Under a Lorentz boost with velocity β along the direction of neutrino propagation, we have that

$$t'_i = \gamma(t_i - \beta x_i), \quad E'_i = \gamma(E_i - \beta p_i), \tag{1.25}$$

where $\gamma = 1/\sqrt{1-\beta^2}$. Then, the phase difference transforms as

$$\begin{aligned}\Delta_{ij} \to \Delta'_{ij} &= E'_i t'_i - E'_j t'_j \\ &= \gamma^2(E_i - \beta p_i)(t_i - \beta x_i) - \gamma^2(E_j - \beta p_j)(t_j - \beta x_j) \\ &= \gamma^2\left[E_i t_i - E_j t_j - \beta(E_i x_i - E_j x_j) - \beta(p_i t_i - p_j t_j) + \beta^2(p_i x_i - p_j x_j)\right].\end{aligned}$$

Well, you should lose hope that this is invariant at this point. But fear not. The key point here is that as long as neutrinos are ultra-relativistic, the invariance can be approximately restored. If this makes you uncomfortable, good. It should. We will revisit this calculation later to make it covariant from the start.
Note that in the ultra-relativistic limit, we can approximate $x_i \simeq t_i$ and $p_i \simeq E_i$, so that

$$\Delta'_{ij} \simeq \gamma^2(1-\beta)^2(E_i t_i - E_j t_j) = \Delta_{ij}, \tag{1.26}$$

which shows that the argument of the sine in the oscillation probability is at least Lorentz invariant in the ultra-relativistic regime.

Three-flavor oscillations. The generalization to n neutrinos is straightforward, but the notation becomes a little more cumbersome. We define the mixing matrix U as the unitary matrix that relates the flavor and mass bases:

$$|\nu_\alpha\rangle = \sum_{i=1}^{n} U_{\alpha i} |\nu_i\rangle, \quad \text{or} \quad |\nu_i\rangle = \sum_{\alpha=1}^{n} U^*_{\alpha i} |\nu_\alpha\rangle. \tag{1.27}$$

The unitarity of U ensures that both bases are orthonormal: $UU^\dagger = U^\dagger U = \mathbb{1}$, which implies

$$\sum_{i=1}^{n} U_{\alpha i} U^*_{\beta i} = \delta_{\alpha\beta}, \quad \text{and} \quad \sum_{\alpha=1}^{n} U^*_{\alpha i} U_{\alpha j} = \delta_{ij}. \tag{1.28}$$

The oscillation probability is then

$$P(\nu_\alpha \to \nu_\beta) = |A_{\nu_\alpha \to \nu_\beta}|^2 = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2. \tag{1.29}$$

Using the same steps as before, we can write the amplitude as

$$A_{\nu_\alpha \to \nu_\beta} = \sum_{i=1}^{n} U^*_{\beta i} U_{\alpha i} e^{-iE_i t}, \tag{1.30}$$

which leads to the oscillation probability

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i,j=1}^{n} U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j} e^{-i\Delta_{ij}}. \tag{1.31}$$

Separating the diagonal and off-diagonal terms, we have

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i=1}^{n} |U_{\beta i}|^2 |U_{\alpha i}|^2 + \sum_{i>j} 2\,\mathrm{Re}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j} e^{-i\Delta_{ij}}\right). \tag{1.32}$$

where we used the simple fact that the $i < j$ and $i > j$ terms are complex conjugates of each other. Squaring the first term of Eq. (1.28) and using the same fact about complex conjugates, we get

$$\delta_{\alpha\beta} = \sum_{i,j=1}^{n} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} = \sum_{i=1}^{n} |U_{\beta i}|^2 |U_{\alpha i}|^2 + \sum_{i>j} 2\,\mathrm{Re}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right). \tag{1.33}$$

Let us use this in the oscillation probability

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i>j} 2\,\mathrm{Re}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}(1 - e^{-i\Delta_{ij}})\right) \tag{1.34}$$

$$= \delta_{\alpha\beta} - \sum_{i>j} 2\,\mathrm{Re}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right)(1 - \cos\Delta_{ij}) \tag{1.35}$$

$$+ \sum_{i>j} 2\,\mathrm{Im}\left(U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}\right)\sin\Delta_{ij}, \tag{1.36}$$

which we can further simplify to

$$\boxed{P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - \sum_{i>j}\left[4\,\mathrm{Re}\left(A^{ij}_{\alpha\beta}\right)\sin^2\frac{\Delta_{ij}}{2} - 2\,\mathrm{Im}\left(A^{ij}_{\alpha\beta}\right)\sin\Delta_{ij}\right],} \tag{1.37}$$

with $A^{ij}_{\alpha\beta} \equiv U^*_{\beta i} U_{\alpha i} U_{\beta j} U^*_{\alpha j}$.

Let us investigate the antineutrino case: $\bar{\nu}_\alpha \to \bar{\nu}_\beta$. The states are related by the complex conjugate of the mixing matrix:

$$|\bar{\nu}_\alpha\rangle = \sum_{i=1}^{n} U^*_{\alpha i} |\bar{\nu}_i\rangle, \quad \text{or} \quad |\bar{\nu}_i\rangle = \sum_{\alpha=1}^{n} U_{\alpha i} |\bar{\nu}_\alpha\rangle. \tag{1.38}$$

So, we can immediately write the oscillation probability for antineutrinos by complex conjugating $A^{ij}_{\alpha\beta}$:

$$P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = \delta_{\alpha\beta} - \sum_{i>j}\left[4\,\mathrm{Re}\left(A^{ij}_{\alpha\beta}\right)\sin^2\frac{\Delta_{ij}}{2} + 2\,\mathrm{Im}\left(A^{ij}_{\alpha\beta}\right)\sin\Delta_{ij}\right]. \tag{1.39}$$

Unsurprisingly, all we really did was flip the sign of the imaginary part.

Helicity states. Where does the neutrino helicity enter in our discussion of oscillations? Well, again because of the ultra-relativistic nature of neutrinos, it is a pretty good assumption to just assume all neutrinos (antineutrinos) are produced as left-helical (right-helical) particles (antiparticles), which are then detected through the Weak interaction as left-chiral (right-chiral) particles (antiparticles). To be clear: neutrinos are produced as different populations of helicity states. But since the Weak interaction only cares about the left-chiral particles (right-chiral antiparticles) components, it produces helicity states that have a larger overlap with left-chiral particles (right-chiral antiparticles).

Let us be a little more precise and define what exactly is produced. Focusing on the *neutrinos* for now, we have a total of six physical states to consider: three mass eigenstates, one of each helicity,

$$\left|\nu_i^-\right\rangle, \quad \left|\nu_i^+\right\rangle, \quad i = 1, 2, 3, \tag{1.40}$$

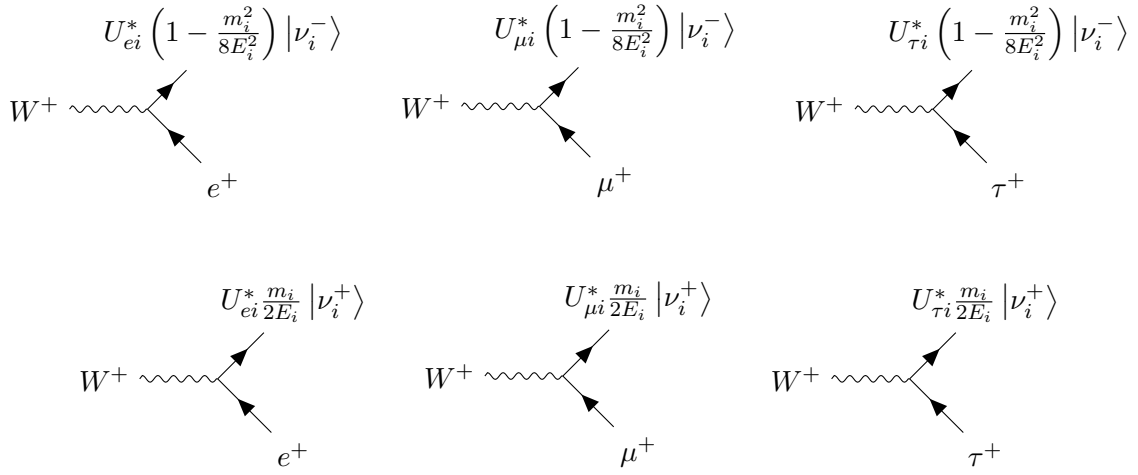
where the $\pm$ superscript indicates the helicity of the state. Imagine we can get a neutrino beam from the decay of W^+ bosons (very hypothetical scenario here). In that case, these six distinct neutrino mass eigenstates are being produced in very specific combinations depending on what charged lepton is produced alongside them. These specific combinations are just the *flavor* states,

$$\left|\nu_\alpha^-\right\rangle = \sum_i^3 U_{\alpha i}^* \left|\nu_i^-\right\rangle, \quad \left|\nu_\alpha^+\right\rangle = \sum_i^3 U_{\alpha i}^* \left|\nu_i^+\right\rangle, \quad \alpha = e, \mu, \tau. \tag{1.41}$$

I remind you that these are not physical states and are not energy eigenstates of the Hamiltonian in vacuum, but they are useful to describe the interactions. So far so good, but the states above have definite helicity, while the Weak interaction only cares about chirality. So, let us go further and ask what is the probability of producing a neutrino of a given helicity, given that the Weak interactions will project our helicity states onto states of definite left-handed chirality with overlap $\langle \nu_i^\pm | \nu_\alpha^L \rangle$. Using the relation between helicity and chirality states for a massive fermion, we have

$$\begin{aligned}
\langle \nu_i^\pm | \nu_\alpha^L \rangle &= \langle \nu_i^\pm | \sum_i^3 U_{\alpha i}^* \left|\nu_i^L\right\rangle && (1.42)\\
&= \langle \nu_i^\pm | \sum_i^3 U_{\alpha i}^* \left(\frac{\sqrt{1+\beta}\left|\nu_i^-\right\rangle + \sqrt{1-\beta}\left|\nu_i^+\right\rangle}{\sqrt{2}} \right) \\
&\simeq \langle \nu_i^\pm | \sum_i^3 U_{\alpha i}^* \left[\left(1 - \frac{m_i^2}{8E_i^2}\right)\left|\nu_i^-\right\rangle + \frac{m_i}{2E_i}\left|\nu_i^+\right\rangle \right] \\
&\simeq \begin{cases} U_{\alpha i}^*(1 - \frac{m_i^2}{8E_i^2}), & \text{for } \nu_i^-, \\ U_{\alpha i}^* \frac{m_i}{2E_i}, & \text{for } \nu_i^+. \end{cases} && (1.43)
\end{aligned}$$

where we use the ultra-relativistic expansion in the third line. In summary, in a neutrino beam obtained from W^+ decay, we have the following completely independent production processes:



Clearly, we can neglect the production of right-helical neutrinos ν_i^+ as it is suppressed by a factor of $(m_i/2E_i)^2$ in the W^+ branching ratios, and even more so, we can neglect the tiny suppression in the production of left-helical neutrinos ν_i^- due to the $m_i^2/8E_i^2$ term. So when we talk about neutrino oscillations, we are really talking about the oscillations of left-helical neutrinos ν_i^- produced in association with a charged lepton of flavor α with amplitude $U^*_{\alpha i}$.

Discrete spacetime symmetries C, P, CP, and T. Let us revisit the behavior of neutrino oscillations under discrete spacetime symmetries. If you recall, charge conjugation, C, swaps particles and antiparticles, so that $\nu_\alpha \leftrightarrow \bar{\nu}_\alpha$, but it does not swap the helicity of the states. So, under C, we have

$$P(|\nu_\alpha^-,\vec{p}\rangle \to \left|\nu_\beta^-,\vec{p}\right\rangle) \xrightarrow{C} P(|\bar{\nu}_\alpha^-,\vec{p}\rangle \to \left|\bar{\nu}_\beta^-,\vec{p}\right\rangle), \tag{1.44}$$

but the left-helical antineutrinos $\bar{\nu}_L^-$ are mostly $\bar{\nu}_L$ states, which do not interact through the Weak interaction. Therefore, what we really care about is the interacting antiparticle states $\bar{\nu}_\alpha^+ \simeq \bar{\nu}_\alpha^R$, which can be obtained with a CP transformation

$$P(|\nu_\alpha^-,\vec{p}\rangle \to \left|\nu_\beta^-,\vec{p}\right\rangle) \xrightarrow{CP} P(|\bar{\nu}_\alpha^+,-\vec{p}\rangle \to \left|\bar{\nu}_\beta^+,-\vec{p}\right\rangle), \tag{1.45}$$

where we unavoidably had to flip the sign of momentum as well. Not to worry, however, as this is exactly what we wanted: if oscillations were developing when a neutrino travels from a point $(\vec{x_1},t_1)$ to $(\vec{x_2},t_2)$ with $\vec{p}$, then under CP they would be developing for an antineutrino traveling from $(-\vec{x_1},t_1)$ to $(-\vec{x_2},t_2)$ with $-\vec{p}$. Our experiment is just had its source-detector setup flipped as well, but everything is consistent.

Under the last discrete symmetry, time-reversal, T, we have

$$P\left(|\nu_\alpha^-,\vec{p}\rangle \to \left|\nu_\beta^-,\vec{p}\right\rangle\right) \xrightarrow{T} P\left(\left|\nu_\beta^-,-\vec{p}\right\rangle \leftarrow |\nu_\alpha^-,-\vec{p}\rangle\right) = P\left(\left|\nu_\beta^-,\vec{p}\right\rangle \to |\nu_\alpha^-,\vec{p}\rangle\right), \tag{1.46}$$

where the flipped arrow of time evolution indicates that we are swapping the initial and final states. Again, this is exactly what we expected: if oscillations were developing when

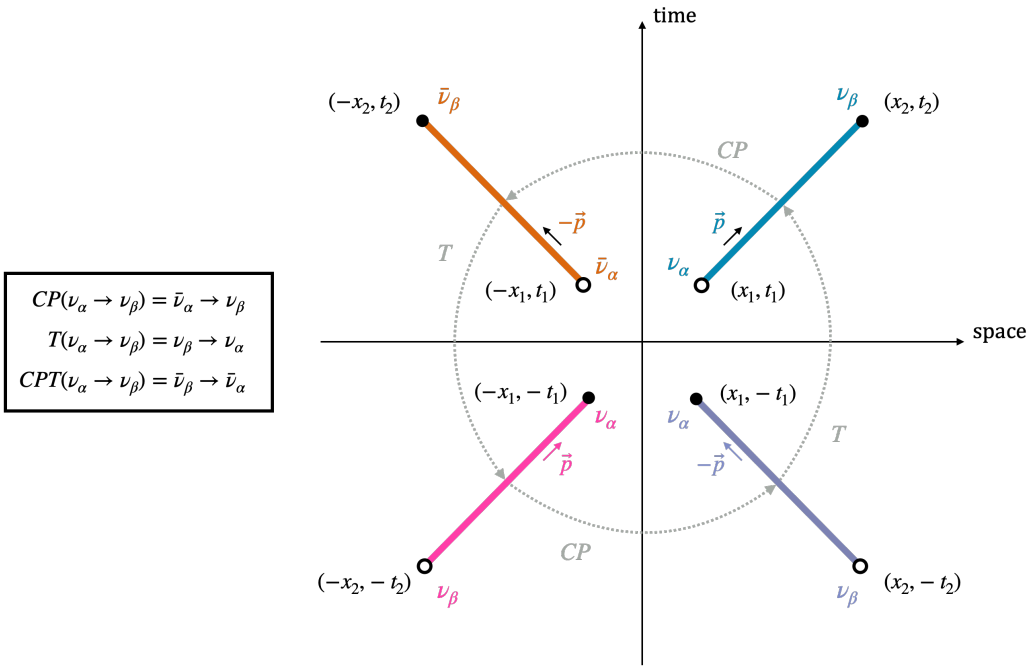


Figure 3: Behavior of neutrino oscillations under discrete spacetime symmetries in a single space dimension.

a neutrino travels from $(\vec{x_1}, t_1)$ to $(\vec{x_2}, t_2)$ with momentum $\vec{p}$, then under T they would be developing for a neutrino traveling from $(\vec{x_2}, -t_2)$ to $(\vec{x_1}, -t_1)$ with momentum $-\vec{p}$.

We can finally combine CP and T to get CPT, which we previously argued should be a good symmetry of any relativistic quantum field theory. Under CPT, we have

$$P\left(\left|\nu_\alpha^-, \vec{p}\right\rangle \to \left|\nu_\beta^-, \vec{p}\right\rangle\right) \xrightarrow{CPT} P\left(\left|\bar{\nu}_\beta^+, -\vec{p}\right\rangle \to \left|\bar{\nu}_\alpha^+, -\vec{p}\right\rangle\right), \tag{1.47}$$

which then should implty that if CPT is conserved, then

$$P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha). \tag{1.48}$$

This can also be seen explicitly by staring at Eq. (1.37). Under CP, we have $A^{ij}_{\alpha\beta} \to (A^{ij}_{\alpha\beta})^*$. Under T, we have can either keep the flavor indices the same and set $t \to -t$ ($\Delta_{ij} \to -\Delta_{ij}$) or we keep t positive and flip the flavor indices ($\alpha \leftrightarrow \beta$), both options correspond to time-reversal. Either way, the combined action of CPT leaves the probability formula invariant. A summary of the behavior of neutrino oscillations under discrete spacetime symmetries is shown in Fig. 3. The key takeaway is that CP and T transformations change the sign of the imaginary part of the oscillation amplitude, while CPT leaves it invariant.

Parameterization. In the 3-neutrino case, the mixing matrix U is a 3×3 unitary matrix called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. Its standard parameteriza-

tion is given by

$$
\begin{aligned}
U &= R_{23}\, R_{13}\, R_{12}\, P \qquad (1.49)\\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\rm CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\rm CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\eta_1} & 0 & 0 \\ 0 & e^{i\eta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\rm CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\rm CP}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} e^{i\eta_1} & 0 & 0 \\ 0 & e^{i\eta_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

where $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$, with $\theta_{12}, \theta_{13}, \theta_{23} \in [0, \pi/2]$ being the three mixing angles, $\delta_{\rm CP} \in [0, 2\pi]$ is the Dirac CP-violating phase, and $\eta_1, \eta_2 \in [0, 2\pi]$ are the two Majorana CP-violating phases.

Note that we have a total of three phases, but that one of them, $\delta_{\rm CP}$, is the only one that appears in outside of a diagonal entry. The other phases η_1 and η_2 are attached to the ν_1 and ν_2 mass eigenstates, and resemble a lot the phase we saw in the 2-neutrino case. As we saw there, these phases do not appear in the oscillation probability, and therefore do not affect neutrino oscillations. It can be shown that for Dirac neutrinos, the phases η_1 and η_2 can be rotated away, while for Majorana neutrinos they are physical parameters that can affect processes where lepton number is violated by two units, such as neutrinoless double beta decay. The additional phase $\delta_{\rm CP}$, however, can never be rotated away or absorbed into the definition of any states and does appear in the oscillation probability. As we will show below, this phase will parameterize the amount of CP violation in neutrino oscillations and is therefore a unique window into the behavior of matter and antimatter. In fact, as far as we know, this is the only feasible way to probe this difference in the lepton sector.

The size of the mixing angles turns out to be quite large, with $\theta_{12} \simeq 33^\circ$, $\theta_{23} \simeq 45^\circ$, and $\theta_{13} \simeq 8.5^\circ$. The latter is the smallest of the three angles, but it is still large enough to be the most-well-measured angle of the three. The CP-violating phase $\delta_{\rm CP}$ is currently poorly constrained, with at least two distinct regions of parameter space allowed at 3σ. The Majorana phases η_1 and η_2 are completely unconstrained (definitely by oscillations, but also by other processes that probe the Majorana nature of neutrinos). The mixing matrix U is therefore very good at scrambling the mass eigenstates into flavor eigenstates, which is why neutrino oscillations are so pronounced.

To see that, we can schematically show the relative size of the entries of the mixing matrix as

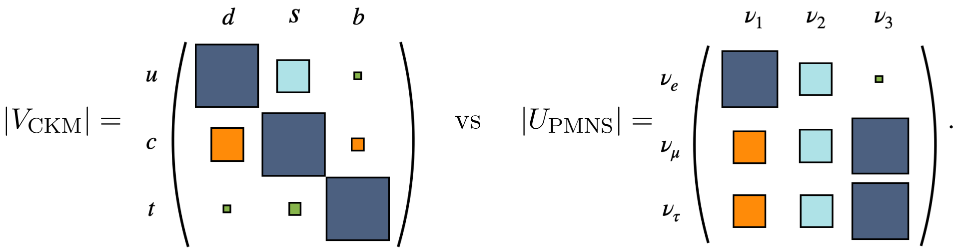
$$
|U| \sim \begin{pmatrix} 0.8 & 0.5 & 0.15 \\ 0.5 & 0.6 & 0.6 \\ 0.3 & 0.6 & 0.7 \end{pmatrix}. \qquad (1.50)
$$

So, when a weak process involves an electron, it produces mostly ν_1, with a significant ν_2 component and a small ν_3 component. The μ and τ processes are more democratic, with all three mass eigenstates contributing more evenly. This is in stark contrast with the quark sector, which also has its mixing matrix called the Cabibbo-Kobayashi-Maskawa (CKM)

matrix. The CKM matrix is very close to the identity because alll mixing angles are very small. Putting the two side-by-side:

$$|V_{\text{CKM}}| \sim \begin{pmatrix} 1 & 0.2 & 0.004 \\ 0.2 & 1 & 0.04 \\ 0.008 & 0.04 & 1 \end{pmatrix}, \quad \text{vs} \quad |U_{\text{PMNS}}| \sim \begin{pmatrix} 0.8 & 0.5 & 0.15 \\ 0.5 & 0.6 & 0.6 \\ 0.3 & 0.6 & 0.7 \end{pmatrix}, \tag{1.51}$$

or in a more visual way,



(1.52)

This is one of the many open questions in particle physics: why is the mixing in the lepton sector so large, while it is so small in the quark sector? In its more general form, this question is part of the so-called **flavor problem**, which asks why the fermion masses and mixings are what they are. The problem in addressing this question is that there are too many theories that explain flavor, often involving new symmetries or dynamics at high energies beyond our reach. Those that are within our reach, however, are often constrained indirectly through observables that depend on flavor, like searches for exotic decays like $\mu \to e\gamma$.

The mass splittings, which dictate the frequency of oscillation are also known to a good precision:

$$\Delta m^2_{21} \simeq 7.4 \times 10^{-5}\ \text{eV}^2, \quad |\Delta m^2_{31}| \simeq 2.5 \times 10^{-3}\ \text{eV}^2. \tag{1.53}$$

The sign of Δm^2_{21} is known, but the sign of Δm^2_{31} is not. As we will see, this is because Δm^2_{21} shows up in the physics of solar neutrinos which propagate through dense matter and can be affected by the matter potential. Note that the oscillation legnths associated with these mass splittings are

$$L^{21}_{\text{osc}} = \frac{4\pi E}{\Delta m^2_{21}} \simeq 3.3 \times 10^4\ \text{km} \left(\frac{E}{\text{GeV}}\right), \quad L^{31}_{\text{osc}} = \frac{4\pi E}{|\Delta m^2_{31}|} \simeq 1.2 \times 10^3\ \text{km} \left(\frac{E}{\text{GeV}}\right). \tag{1.54}$$

So, for neutrinos with energies around 1 GeV, the oscillation length associated with Δm^2_{31} is long, but still within reason. It means we can send GeV-scale neutrino beams from pion decay, ν_μ and $\bar{\nu}_\mu$, through the Earth to see them oscillate to $\nu_\mu \to \nu_e$ or $\bar{\nu}_\mu \to \bar{\nu}_e$ as well as $\nu_\mu \to \nu_\tau$ or $\bar{\nu}_\mu \to \bar{\nu}_\tau$. The oscillation length associated with Δm^2_{21}, however, is about $30,000$ km, which larger than the diameter of the Earth. The solution here is to go to lower energies, in the MeV range, where the oscillation length is about 30 km, which is much more manageable. However, in that case, we only know how to produce electron

neutrinos, which in turn are not energetic enough to produce muons, so we only have access to disappearance channels such as $\nu_e \to \nu_e$ or $\bar{\nu}_e \to \bar{\nu}_e$. Lukily, solar neutrinos are in this energy range, and we can use them to probe Δm^2_{21} in a a way a very complementary way thanks to the matter effects they experience while propagating out of the Sun.

CP violation. It is useful to define the CP asymmetry

$$\mathcal{A}^{\alpha\beta}_{\rm CP} = P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = 4 \sum_{i>j} \text{Im}\left(A^{ij}_{\alpha\beta}\right) \sin \Delta_{ij}. \tag{1.55}$$

In fact, under time-reversal, T, the amplitudes also flip sign, $A^{ij}_{\alpha\beta} \to -A^{ij}_{\alpha\beta}$, so this asymmetry is also a measure of T violation (at least in the vacuum case we are considering so far). If the imaginary part of the oscillation amplitude is non-zero, then CP violation is manifest and its amount can be measured by comparing the oscillation probabilities of neutrinos and antineutrinos.

It turns out that the quantity $\text{Im}\left(A^{ij}_{\alpha\beta}\right)$ is proportional to a universal quantity called the Jarlskog invariant, J, which is independent of the choice of flavor and mass indices:

$$\boxed{\text{Im}\left(A^{ij}_{\alpha\beta}\right) = \text{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) = \pm J, \quad \text{with} \quad J = c_{12} c^2_{13} c_{23} s_{12} s_{13} s_{23} \sin\delta,} \tag{1.56}$$

where the sign depends on the specific choice of flavor and mass indices.

Jarlskog invariant: The Jarlskog invariant is a measure of CP violation in the Standard Model, originally introduced in the context of quark mixing by Cecilia Jarlskog [1]. It is quite often the case that the individual phase of a fermion fields is not physical, as we could have just absorbed such a phase into the definition of the field without changing the physics. The Jarlskog invariant, however, is a combination of phases of fermion fields that turns out to not only be impossible to rotate away but also to be invariant under any choice of fermion field basis, making it a physically meaningful quantity. In the context of quark or lepton mixing, the Jarlskog invariant quantifies the amount of CP violation in the respective sector.

Show that $\text{Im}\left(A^{ij}_{\alpha\beta}\right) = \text{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j}\right) = \pm J$ by explicit calculation using Eq. (1.49). Convince yourself that the Majorana phases η_1 and η_2 do not enter into this result.

The CP asymmetry can be either positive or negative, depending on the sign of $\sin\delta_{\rm CP}$ and the sign of $\sin\Delta_{ij}$. However, you can show that the size of the asymmetry is bounded by

$$\mathcal{A}^{\alpha\beta}_{\rm CP} = \pm 4J \left(\sin\Delta_{31} + \sin\Delta_{21} - \sin\Delta_{32}\right) \simeq \pm 0.132 \times \sin\delta_{\rm CP}, \tag{1.57}$$

where $\pm$ depends on the choice of flavor indices ($\alpha \neq \beta$), and we used the best-fit values of the mixing angles to get the numerical value. So to measure CP violation in neutrino oscillations, we need to be able to measure an asymmetry that is a $\mathcal{O}(10\%)$ effect or smaller.

CP violation requires 3 flavors. Let us now prove that CP violation requires at least three neutrino flavors. Consider the 2-neutrino case with ν_e and ν_μ, for instance. Also, define the oscillation probabilities

$$P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta), \quad \text{and} \quad \bar{P}_{\alpha\beta} \equiv P(\bar{\nu}_\alpha \to \bar{\nu}_\beta). \tag{1.58}$$

If the time evoution is unitary, then the oscillation probabilities must satisfy

$$P_{ee} + P_{e\mu} = 1, \quad P_{\mu e} + P_{\mu\mu} = 1, \tag{1.59}$$

and similarly for antineutrinos,

$$\bar{P}_{ee} + \bar{P}_{e\mu} = 1, \quad \bar{P}_{\mu e} + \bar{P}_{\mu\mu} = 1. \tag{1.60}$$

This is just a statement that the neutrino must be found in one of the two flavor states. The CP asymmetry is then

$$\mathcal{A}_{\rm CP}^{e\mu} = P_{e\mu} - \bar{P}_{e\mu} = (1 - P_{ee}) - (1 - \bar{P}_{ee}), \tag{1.61}$$

however, CPT invariance implies that

$$P_{\alpha\beta} = \bar{P}_{\beta\alpha} \implies P_{ee} = \bar{P}_{ee}, \tag{1.62}$$

which in turn implies that

$$\mathcal{A}_{\rm CP}^{e\mu} = 0. \tag{1.63}$$

Therefore, CP violation is not possible in the 2-neutrino case. In the 3-neutrino case, we would have instead

$$P_{ee} + P_{e\mu} + P_{e\tau} = 1, \quad P_{\mu e} + P_{\mu\mu} + P_{\mu\tau} = 1, \quad P_{\tau e} + P_{\tau\mu} + P_{\tau\tau} = 1, \tag{1.64}$$

, and by using the CPT invariance on the disappearance channels, we would have found that

$$\mathcal{A}_{\rm CP}^{e\mu} = P_{e\mu} - \bar{P}_{e\mu} = -P_{e\tau} + \bar{P}_{e\tau} = -\mathcal{A}_{\rm CP}^{e\tau}, \tag{1.65}$$

which is not necessarily zero.

References

[1] C. Jarlskog, *Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation*, *Phys. Rev. Lett.* **55** (1985) 1039.