1 Neutrino masses and the Seesaw Mechanism

In general, we can collect all different neutrino mass types into a matrix to describe how mass terms connect different neutrino flavor states. In this language, a mass is just an interaction that connects different neutrino states. Using the bra-ket notation, masses are then part of our Hamiltonian and will act on neutrino states to change their type. There are Dirac mass terms that connect left-handed and right-handed neutrinos, and there are Majorana mass terms that connect neutrinos to antineutrinos. We typically collect all such "interactions" into mass matrix that has left-handed type particles on the left-hand side and right-handed states type particles on the right. For a single generation of neutrinos (think $\nu_L = \nu_e$, for instance), the most general mass matrix that we can write is

$$\begin{aligned}\mathcal{H}_{\text{int}} &= m_D \left|\nu_L\right\rangle \left\langle\nu_R\right| + m_D \left|\bar{\nu}_L\right\rangle \left\langle\bar{\nu}_R\right| + \mu_L \left|\nu_L\right\rangle \left\langle\bar{\nu_L}\right| + \mu_R \left|\nu_R\right\rangle \left\langle\bar{\nu_R}\right| + \text{h.c.} \\ &= \begin{pmatrix}\left|\nu_L\right\rangle & \left|\bar{\nu_L}\right\rangle\end{pmatrix} \begin{pmatrix}\mu_L & m_D \\ m_D & \mu_R\end{pmatrix} \begin{pmatrix}\left\langle\bar{\nu_R}\right| \\ \left\langle\nu_R\right|\end{pmatrix} + \text{h.c.}\end{aligned} \tag{1.1}$$

where $|\nu_L\rangle$ is the usual left-handed neutrino and $|\nu_R\rangle$ is a new right-handed neutrino (non-interacting). Recall that because of CPT invariance, ν_L and ν_R must come with their respective antiparticles, $\bar{\nu}_L$ and $\bar{\nu}_R$. Here, m_D is a Dirac mass term that can come from the Higgs mechanism, μ_L is a Majorana mass term for the active (Weakly-interacting) neutrino (this needed two Higgs insertions), and μ_R is a Majorana mass term for the sterile neutrino. The parameters m_D, μ_L, and μ_R all have dimensions of mass.

The term "h.c." stands for Hermitian conjugate. In this case, it ensures that the theory is Hermitian (and therefore conserves probability). In practice, it means whatever I can do to the left-handed neutrino states, I must also be able do to the right-handed neutrino states. We can write the hermitian conjugate term by transposing the whole expression and taking complex conjugates of all parameters (we assume them to be real here anyway, so this last step does not do anything). Using that $(|\psi\rangle)^\dagger = \langle\psi|$ and $(\langle\psi|)^\dagger = |\psi\rangle$, we get

$$\begin{pmatrix}\left|\bar{\nu}_R\right\rangle & \left|\nu_R\right\rangle\end{pmatrix} \begin{pmatrix}\mu_L & m_D \\ m_D & \mu_R\end{pmatrix} \begin{pmatrix}\left\langle\nu_L\right| \\ \left\langle\bar{\nu}_L\right|\end{pmatrix} \tag{1.2}$$

So it is just the same expression after swapping states for their CPT-conjugate ones:

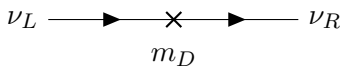
$$\left|\nu_L\right\rangle \overset{\text{CPT}}{\Longleftrightarrow} \left|\bar{\nu}_R\right\rangle, \quad \left|\bar{\nu}_L\right\rangle \overset{\text{CPT}}{\Longleftrightarrow} \left|\nu_R\right\rangle. \tag{1.3}$$

Recall that this means that they arise from the same quantum field.

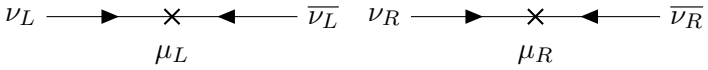
This is somewhat vague and only an effective description of the physics of mass, but it works quite well for most purposes. You can think of something like $m_D \left|\nu_L\right\rangle \left\langle\nu_R\right|$ as shorthand for an interaction Hamiltonian that takes a right-handed neutrino state $\left|\nu_R\right\rangle$ and turns it into a left-handed neutrino state $\left|\nu_L\right\rangle$ with amplitude m_D, or vice versa for the Hermitian conjugate term. In other words, the amplitude for

$$A(\nu_L \to \nu_R) = \left\langle\nu_R\right| \mathcal{H}_{\text{int}} \left|\nu_L\right\rangle = \left\langle\nu_R\right| \Big(m_D \left|\nu_R\right\rangle \left\langle\nu_L\right| \Big) \left|\nu_L\right\rangle \tag{1.4}$$

In terms of Feynman diagrams, you can represent this mass term as a fermion line going from ν_L to ν_R with a mass insertion m_D:



and similarly for the Majorana mass terms



More formally, the mass terms are described by the underlying quantum fields in our Lagrangian. In terms of 4-component spinor fields, the Dirac mass term comes from $\mathcal{L}_{\text{Dirac}} = -m_D \bar{\nu_L} \nu_R + \text{h.c.}$ and the Majorana mass terms come from $\mathcal{L}_{\text{Majorana}} = -\frac{1}{2}\mu_L \nu_L^T C \nu_L - \frac{1}{2}\mu_R \nu_R^T C \nu_R + \text{h.c.}$, where C is the charge-conjugation matrix.

Find the rotation angle θ that diagonalizes this matrix. Now find the eigenvalues of this matrix. Expand them in the following two limits:

1. $\mu_L, \mu_R = 0$ (the Dirac limit)
2. $\mu_L, \mu_R \ll m_D$ (the pseudo-Dirac/quasi-Dirac limit)
3. $\mu_R \gg m_D$ (the seesaw limit, where all particles are of the Majorana type). You can either keep μ_L general (assuming $\mu_R \gg \mu_L$) or set it to zero.

Using the very useful formula for diagonalizing a 2×2 symmetric matrix, we get

$$\tan 2\theta = \frac{2m_D}{\mu_R - \mu_L}. \tag{1.5}$$

Note that this formula is valid for any values of the parameters. The point here is that the flavor basis (the weak-interaction eigenstates) is rotated by an angle θ to get to the mass basis (the propagating eigenstates):

$$D = \text{diag}(m_1, m_2) = R(\theta)^T . M . R(\theta) \text{ with } R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1.6}$$

To find the eigenvalues of the mass matrixk, follow a few different procedures. You can perform the rotation above by hand, or you can solve the characteristic polynomial $\det(M - \lambda I) = 0$, or you can use the fact that for a 2×2 matrix, the eigenvalues are given by

$$m_{1,2} = \frac{1}{2}\left(\mathrm{Tr}(M) \pm \sqrt{\mathrm{Tr}(M)^2 - 4\det(M)}\right). \tag{1.7}$$

In our case, we have $\mathrm{Tr}(M) = \mu_L + \mu_R$ and $\det(M) = \mu_L\mu_R - m_D^2$. Therefore, the eigenvalues are

$$m_{1,2} = \frac{1}{2}\left(\mu_L + \mu_R \pm \sqrt{(\mu_L - \mu_R)^2 + 4m_D^2}\right). \tag{1.8}$$

This relation comes from the fact that

$$\mathrm{Tr}(M) = \mathrm{Tr}\big(R^T D R\big) = \mathrm{Tr}\big(DRR^T\big) = \mathrm{Tr}(D) = m_1 + m_2, \tag{1.9}$$

and

$$\det(M) = \det\big(R^T D R\big) = \det\big(R^T\big)\det(D)\det(R) = \det(D)\det\big(R^T R\big) = m_1 m_2. \tag{1.10}$$

These are two equations with two unknowns, so you can solve for $m_{1,2}$ by solving the underlying quadratic equation. This is analogous to solving the characteristic polynomial, but it can be faster. For instance, for a zero-determinant matrix, you immediately know one eigenvalue is zero, so you can get the other one from the trace. Recall, such matrices are singular and, therefore, non-invertible. More importantly, their rank is less than the number of dimensions. For a square matrix of size $n \times n$, this means that $\mathrm{rank}(M) < n$, that is, the number of eigenvalues that are non-zero is less than n.

Now, let's define the following useful parameters:

$$\mu = \mu_R + \mu_L, \quad \Delta\mu = \mu_R - \mu_L. \tag{1.11}$$

In terms of these, the eigenvalues can be written as

$$m_{1,2} = \frac{1}{2}\left(\mu \pm \sqrt{\Delta\mu^2 + 4m_D^2}\right). \tag{1.12}$$

Now, we can expand these eigenvalues in the different limits.

Dirac In the Dirac limit, $\mu_L, \mu_R = 0$, we have $\mu = 0$ and $\Delta\mu = 0$, so

$$\boxed{m_{1,2} = \pm m_D.} \tag{1.13}$$

This is saying that these two neutrino states combine into a Dirac fermion of mass m_D. The fact that we get a negative mass for one of the states might seem worrisome at first, but it just means that the two states have opposite complex phases. Physically, that means they have opposite CP phases (see below).

A negative mass fermion?! Not to worry, this is just telling us that a Dirac fermion is made up of two degenerate Majorana fermions with the same mass, but opposite CP phases. For those familiar with QFT, you can see that by writing the mass term in the Lagrangian:

$$\mathcal{L} \supset -\frac{1}{2}\left(m_1 \nu_1^T C \nu_1 + m_2 \nu_2^T C \nu_2\right) + \text{h.c.} = -\frac{1}{2} m_D \left(\nu_1^T C \nu_1 - \nu_2^T C \nu_2\right) + \text{h.c.}$$

Recall that a Majorana mass term is of the form $\psi^T C \psi = \bar{\psi} \psi^c$ where $\psi^c = C \bar{\psi}^T$. Here ψ is a 4-component spinor and C is the charge-conjugation matrix (in the Dirac basis, $C = i\gamma^2\gamma^0$). Now, defining $\nu = (\nu_1 + i\nu_2)/\sqrt{2}$ and $\bar{\nu} = (\nu_1 - i\nu_2)/\sqrt{2}$, we get

$$\mathcal{L} \supset -m_D \bar{\nu} \nu,$$

which is the usual Dirac mass term. Note that we are free to redefine the phases of the fields to make masses positive. For example, we could redefine $\nu_2 \to i\nu_2$ to make both masses positive. This means that the phase of ν_2 is different from that of ν_1 by $\pi/2$, which is what gives us the opposite CP phases.

Quasi-Dirac In the quasi-Dirac limit, $\mu_L, \mu_R \ll m_D$, we have $\mu \ll m_D$ and $\Delta\mu \ll m_D$. Let us rewrite the general expression for the eigenvalues as

$$m_{1,2} \approx m_D \left(\frac{\mu}{2m_D} \pm \sqrt{1 + \left(\frac{\Delta\mu}{2m_D}\right)^2} \right). \tag{1.14}$$

We now treat $\mu/(2m_D)$ and $\Delta\mu/(2m_D)$ as small parameters and expand the square root to first order using $\sqrt{1+x} \approx 1 + x/2$ for $x \ll 1$.

$$m_{1,2} \approx m_D \left(\frac{\mu}{2m_D} \pm \left(1 + \frac{1}{2}\left(\frac{\Delta\mu}{2m_D}\right)^2\right) \right) = \pm m_D + \frac{\mu}{2} \pm \frac{(\Delta\mu)^2}{8m_D}. \tag{1.15}$$

In reality, the last term is negligible compared to the first two, so we can write

$$\boxed{m_{1,2} \simeq \pm m_D + \frac{\mu}{2}.} \tag{1.16}$$

Recalling that the overall minus sign is just a phase, this expression is telling us that we have two *nearly-degenerate* Majorana neutrinos with masses close to m_D, split by a total amount μ.

Seesaw In the seesaw limit, $\mu_R \gg m_D$. Initially, let us set $\mu_L = 0$ for simplicity. In that case, $\mu = \mu_R$ and $\Delta\mu = \mu_R$ and the eigenvalues become

$$m_{1,2} = \frac{\mu_R}{2} \pm \frac{1}{2}\sqrt{\mu_R^2 + 4m_D^2}. \tag{1.17}$$

Expanding in the small parameter m_D/μ_R, we get

$$\boxed{m_1 \simeq -\frac{m_D^2}{\mu_R}, \quad m_2 \simeq \mu_R.} \tag{1.18}$$

This is the seesaw mechanism at work: one neutrino becomes very heavy, with mass $\sim \mu_R$, while the other becomes very light, with mass $\sim m_D^2/\mu_R$.

For completeness, let us re-introduce μ_L. As long as it remains much smaller than μ_R, the expansion still holds. The eigenvalues as

$$m_{1,2} = \frac{\mu_R}{2} + \frac{\mu_L}{2} \pm \frac{\mu_R}{2}\sqrt{\left(1 - \frac{\mu_L}{\mu_R}\right)^2 + 4\left(\frac{m_D}{\mu_R}\right)^2} \tag{1.19}$$

$$\simeq \frac{\mu_R}{2} + \frac{\mu_L}{2} \pm \frac{\mu_R}{2}\left(1 - \frac{\mu_L}{\mu_R} + \frac{\mu_L^2 + 4m_D^2}{2\mu_R^2}\right) \tag{1.20}$$

The eigenvalues are then

$$m_1 \simeq \mu_L - \frac{\mu_L^2 + 4m_D^2}{4\mu_R}, \tag{1.21}$$

$$m_2 \simeq \mu_R + \frac{\mu_L^2 + 4m_D^2}{4\mu_R}. \tag{1.22}$$

The only difference is that the light neutrino mass gets a small correction from μ_L (with the opposite sign!). In practice, this contribution is usually neglected because μ_L is incompatible with the symmetries of the SM (electroweak gauge invariance) and is expected to be very small compared to the other terms.

b In the seesaw limit, call the physical basis states ν_ℓ and ν_h, where ν_h is the heavy neutrino, sometimes called a heavy neutral lepton (HNL). What is the relation between the weak-interaction eigenstate ν and these mass eigenstates in terms of θ? Which state ν_ℓ or ν_h will interact more strongly with the rest of the SM particles?

Using the rotation in Eq. (**??**), we have

$$|\nu_L\rangle = \cos\theta\,|\nu_\ell\rangle + \sin\theta\,|\nu_h\rangle\,, \quad |\nu_R\rangle = -\sin\theta\,|\nu_\ell\rangle + \cos\theta\,|\nu_h\rangle\,. \tag{1.23}$$

The mixing angle in the seesaw limit is small, namely,

$$\theta \simeq \frac{m_D}{\mu_R}, \tag{1.24}$$

so that we can write

$$|\nu_L\rangle \simeq |\nu_\ell\rangle + \theta\,|\nu_h\rangle\,, \quad |\bar{\nu}_R\rangle \simeq -\theta\,|\nu_\ell\rangle + |\nu_h\rangle\,. \tag{1.25}$$

Multiplying things out, you should see that the light mass state is the one that most closely resembles ν_L (the interacting neutrino state), and therefore, is the one that interacts more with SM particles (through the Weak force).

c Identify ν_ℓ state with the neutrinos we observe in the laboratory and take $m_{\nu_\ell} \sim 0.05$ eV. Recalling that m_D comes from the Higgs mechanism, $m_D = yv/\sqrt{2}$ with $v = 246$ GeV and y a coupling constant, we will take m_D to be in the rage $[m_e, m_t] = [511\text{ keV}, 176\text{ GeV}]$, corresponding to $y \sim [3\times10^{-6}, 1]$. What are the values of μ that you need in order to obtain the assumed value for m_{ν_ℓ}? What values of the mixing angle do these correspond to?

Putting in the numbers, we get roughly, $\mu_R \sim [10^3 \text{ GeV}, 10^{14}\text{GeV}]$ and $\theta \sim [10^{-7}, 10^{-12}]$. You can see that even if the Dirac mass term is as small as the electron one, the required Majorana mass (and therefore the mass of the propagating state ν_h) will be very large and will be quickly untestable as you raise m_D. All relevant cross sections and decay rates for detecting ν_h, in fact, scale as θ^2, so that makes these particles very hard to observe in the laboratory, even if we had the energy to produce them. We say then, that ν_h has *weaker-than-Weak* interactions, meaning that the only part of ν_h that does interact (a fraction θ of its superposition) interacts only through the Weak force.