## 1 Phase space

In this section we will delve a bit deeper into the counting of the final states phase space factors. In doing so, we will provide half of the ingredients needed to calculate decay rates and cross sections, which are the main observables in particle physics. The other half is the amplitude, which we will not calculate explicitly in this course.

Recall that in Fermi's Golden Rule, the transition rate from an initial state $|i\rangle$ to a final state $|f\rangle$ is given by

$$d\Gamma = |M_{i\to f}|^2 2\pi\delta(E_f - E_i)\rho(E_f)dE_f \tag{1.1}$$

where in our perturbative expansion

$$M_{i\to f} = g\,\langle f|\,\hat{H}_{\text{int}}\,|i\rangle + g^2 \sum_n \frac{\langle f|\,\hat{H}_{\text{int}}\,|n\rangle\,\langle n|\,\hat{H}_{\text{int}}\,|i\rangle}{E_i - E_n} + \mathcal{O}(g^3) \tag{1.2}$$

is the amplitude for the incoming state $|i\rangle$ to scatter into the final state $|f\rangle$ as it interacts with a region where the interaction Hamiltonian $\hat{H}_{\text{int}}$ (or a potential $V(x)$) is important. The factor $\rho(E_f)$ is the density of final states with energies between $E_f$ and $E_f + dE_f$.

While in non-relativistic QM, the potential was the thing that our particles scattered on, in particle physics, we care about the scattering between particles. Previously, we modeled electron scattering on a heavy nucleus of charge $Z$ as the scattering of the electron on the Coulomb potential $V(r) = -\frac{Ze^2}{r}$. Now, we will replace this notion by the scattering between particles themselves, which enables us to describe the behavior of both particles involved in the reaction: the projectile electron and the target nucleus. And because in a relativistic theory particles can be created and destroyed, we will have to account for the possibility that the number of particles in the initial and final states may be different. The amplitude $\mathcal{M}$, therefore, should be calculated for the transition of some initial state made of multiple particles of momenta $p_1, p_2, \ldots$, namely $|i\rangle = |p_1, p_2, \ldots\rangle$, into some final multi-particle state $|f\rangle = |p'_1, p'_2, \ldots\rangle$. If you have a single particle in the initial state, you are describing a decay process, while if you have two or more particles in the initial state, you are describing a scattering process. It is rare that one would encounter more than two incoming states, but in principle our formalism will also handle those cases as well.

As we account for this multi-particle scattering, it is also important to note that the above formula for Fermi's Golden rule is not invariant under Lorentz transformations. Going forward, we will require it to be explicitly invariant, which will force us to modify the phase space measure, $\rho(E_f)dE_f$, as well as the normalization of our states $|i\rangle$ and $|f\rangle$. Initially, our plane-waves in a box of volume $V$ were all normalized according to

$$\int_V d^3x |\psi(\vec{x})|^2 = \int_V d^3x \left|\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{V}}\right|^2 = 1. \tag{1.3}$$

This is clearly bad, as the quantity $\int d^3x$ is not Lorentz-invariant. Another way to see why it fails is to note that the volume $V$ is not necessarily the same as the volume in the integration measure if we boost our system in any direction, since the box will contract in the direction of the boost and the cancellation between volumes is not guaranteed. Hence, the normalization of our states will depend on what frame we defined $V$.

**Lorentz-invariant wavefunction normalization.** To recover Lorentz invariance, we will work with wavefunctions that are normalized to $2E$ instead of 1. That is, we want to work with wavefunctions $\psi'(\vec{x}) = \sqrt{2E}\psi(\vec{x})$ such that the state

$$\left|\psi'\right\rangle = \sqrt{2E}\left|\psi\right\rangle, \tag{1.4}$$

is normalized according to

$$\int_V d^3x|\psi'(\vec{x})|^2 = \int_V d^3x\left|\sqrt{2E}\psi(\vec{x})\right|^2 = 2E\int_V d^3x\left|\frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{V}}\right|^2 = 2E, \tag{1.5}$$

where $E$ is the energy of the particle described by the wavefunction $\psi(\vec{x})$. It may seem surprising that this condition is invariant. After all, there is an explicit factor of energy on the right-hand side. However, you can convince yourself that it is the case because $d^3x$ transforms as $d^3x \to d^3x/\gamma$ under a boost in, say, the $z$ direction, while $E \to \gamma E$. Therefore, the cancellation between the volume factor in the integral and the normalization of the wavefunction is guaranteed in any frame.

Adjusting the normalization for all particles using Eq. (1.4) and explicitly pulling out the factor of volume $V$, we have

$$|f\rangle = |p_1, p_2, \ldots, p_N\rangle = \frac{1}{\sqrt{2E_1V}\sqrt{2E_2V}\cdots\sqrt{2E_NV}}\left|f'\right\rangle = \prod_{i=1}^{N}\frac{1}{\sqrt{2E_iV}}\left|f'\right\rangle, \tag{1.6}$$

with $|f'\rangle$ normalized to $\langle f'|f'\rangle = 2E_1 2E_2 \cdots 2E_N$. Similarly, for the initial state with $M$ particles, we have

$$|i\rangle = |p_1, p_2, \ldots, p_M\rangle = \prod_{i=1}^{M}\frac{1}{\sqrt{2E_iV}}\left|i'\right\rangle, \tag{1.7}$$

with $|i'\rangle$ normalized to $2E_1 2E_2 \cdots 2E_M$, such that now we are interested in the transition amplitude

$$M_{i\to f} = \langle f|\,\hat{H}_{\text{int}}\,|i\rangle = \prod_{i=1}^{N}\frac{1}{\sqrt{2E_iV}}\prod_{j=1}^{M}\frac{1}{\sqrt{2E_jV}}M_{i'\to f'}, \tag{1.8}$$

where the amplitude $M_{i'\to f'} = \langle f'|\,\hat{H}_{\text{int}}\,|i'\rangle$ is now calculated using the normalized states $|i'\rangle$ and $|f'\rangle$ (up to factors of volume $V$ that we pulled out explicitly).

**Lorentz-invariant phase space.** Having addressed the states, we now look at their density in phase space, $\rho(E_f)dE_f$. We should be able to count the number of states in a box of volume $V$ with energies between $E_f$ and $E_f + dE_f$ by doing the following:

$$\rho(E_f)dE_f = \frac{V}{(2\pi)^3}d^3p_f. \tag{1.9}$$

But now we have multiple particles in the final state, so we should really write one such factor for each independent set of momenta in the final state. If we have $N$ particles in the final state, energy-momentum conservation will impose a constraint on one of the particles, but all $N-1$ others are independent, such that the density of states is given by

$$\rho(E_f)dE_f \longrightarrow V^{N-1}\prod_{k=1}^{N-1}\frac{d^3\vec{p}_k}{(2\pi)^3} \longrightarrow (2\pi)^3 V^{N-1}\prod_{k=1}^{N}\frac{d^3\vec{p}_k}{(2\pi)^3}\delta\left(\vec{p}_i-\sum_{\ell=1}^{N}\vec{p}_\ell\right), \tag{1.10}$$

where $\vec{p}_i$ is the total momentum of the initial state, and we have introduced a delta function to enforce momentum conservation by using the fact that

$$\int d^3\vec{p}_N\delta\left(\vec{p}_i-\left(\sum_{k}^{N-1}\vec{p}_k\right)-\vec{p}_N\right)=1. \tag{1.11}$$

Note that in Eq. (1.1) we had a delta function $\delta(E_f-E_i)$ that enforces energy conservation. However, in a relativistic theory, we should really be enforcing energy-momentum conservation, which is precisely what the product of delta functions will do:

$$\delta^{(4)}(p_i-p_f)=\delta(E_i-E_f)\delta(p_i^x-p_f^x)\delta(p_i^y-p_f^y)\delta(p_i^z-p_f^z)=\delta(E_f-E_i)\delta^{(3)}(\vec{p}_f-\vec{p}_i), \tag{1.12}$$

where we use the superscrpit (4) to denote a four-dimensional delta function.

**Updating Fermi's Golden Rule.** Pulling all these factors together in Fermi's Golden Rule, we have

$$d\Gamma = |M_{i\to f}|^2 2\pi\delta(E_f-E_i)\rho(E_f)dE_f \tag{1.13}$$

$$\longrightarrow |M_{i'\to f'}|^2(2\pi)^4\delta^{(4)}(p_i-p_f)\,(V)^{N-1}\prod_{k=1}^{M}\frac{1}{2E_kV}\prod_{\ell=1}^{N}\frac{d^3\vec{p}_\ell}{2E_\ell(2\pi)^3V}, \tag{1.14}$$

$$= \frac{|M_{i'\to f'}|^2}{V^{M+1}}(2\pi)^4\delta^{(4)}(p_i-p_f)\prod_{i=1}^{N}\frac{1}{2E_k}\prod_{\ell=1}^{M}\frac{d^3\vec{p}_\ell}{(2\pi)^3 2E_\ell}, \tag{1.15}$$

where in addition to Eq. (1.10) we have used the fact that the amplitude $M_{i'\to f'}$ is calculated using the normalized states $|i'\rangle$ and $|f'\rangle$, which introduced factors of $1/\sqrt{2EV}$ for each particle in the initial and final states, respectively.[1]

[1] I should say, this still does not quite look very relativistic. The initial particles show up with explicit factors of $1/2E$ in the denominator, which is a bit odd. This too, can be fixed into a more relativistic form, but it is very much overkill for now. After all, our initial states usually *do* break Lorentz invariance. They come from particle beams of specific direction or constitute particle targets at rest. So this form is enough for now. In applications like cosmology, where the initial states are in a thermal bath, one can do better and write the initial state in a more relativistic way since the initial states are isotropically and homogeneously distributed in space.

Note that not all factors of volume $V$ cancel out. There is an important story here regarding our Hamiltonian. In a relativistic theory, we deal not with a Hamiltonian but with a Hamiltonian density $\mathcal{H}(x) \sim H/V$. In more precise language, the Hamiltonian density is defined by $H = \int d^3x \mathcal{H}(x)$. This is not surprising in a relativistic theory as all four space-time components should be treated on the same footing.

With this in mind, we can redefine our interaction Hamiltonian $H_{\text{int}} = \int d^3x \mathcal{H}_{\text{int}}(x)$, such that the factor of $1/V$ is absorbed in the definition of the interaction Hamiltonian density $\hat{\mathcal{H}}_{\text{int}}$. Schematically, we can replace $\hat{H}_{\text{int}} V^{-1} \to \hat{\mathcal{H}}_{\text{int}}$, or at the level of the amplitude $M_{i' \to f'} V^{-1} \to \mathcal{M}_{i' \to f'} = \langle f' | \hat{\mathcal{H}}_{\text{int}} | i' \rangle$. Cursive $\mathcal{M}$ is often used to denote the amplitude calculated with the Hamiltonian density, while straight $M$ is used when the amplitude is calculated with the Hamiltonian itself. These have different mass dimensions, which is how we keep track of the volume factors explicitly.

Absorbing a factor of $1/V^2$ into the square of the transition amplitude to make use of the Hamiltonian density, $\hat{\mathcal{H}}_{\text{int}}$, we find the true final expression for the transition rate

$$\boxed{d\Gamma = \frac{|\mathcal{M}_{i' \to f'}|^2}{V^{M-1}} (2\pi)^4 \delta^{(4)}(p_i - p_f) \prod_{k=1}^{M} \frac{1}{2E_k} \prod_{\ell=1}^{N} \frac{d^3\vec{p}_\ell}{(2\pi)^3 2E_\ell}.} \tag{1.16}$$

The volume factor remains with a power of $M - 1$, where $M$ is the number of particles in the initial state. For example, for a decay process where $M = 1$, the volume factor disappears completely, as it should. For a $2 \to N$ scattering process, we have $M = 2$, such that a single factor of $1/V$ remains, which is expected as the scattering rate should be proportional to the density of incoming particles, which is $1/V$ for our particle inside a box of volume $V$. For $M > 2$, the interpretation is less clear, but such processes are rare in particle physics. That being said, it makes sense that for every additional initial state we require, the rate should be proportional to the product of the densities of each additional particle, which is $1/V$ for each additional particle.

Show that $\frac{d^3\vec{p}}{2E}$ is invariant under Lorentz transformations.

What are the units of the amplitude $\mathcal{M}$? To answer this question, we need to know the units of the decay rate $[\Gamma] = \text{GeV}$, as well as the units of the phase space measure. The phase space measure is given by

$$d\Pi_f = (2\pi)^4 \delta^{(4)}(p_i - p_f) \prod_{j=1}^{N} \frac{d^3\vec{p}_j}{(2\pi)^3 2E_j}.$$

The delta function has units of inverse energy to the fourth power, $[\delta^{(4)}] = \text{GeV}^{-4}$. You can convince yourself of this by noting that the delta function is defined by the property $\int dx\delta(x) = 1$, such that $[\delta(x)] = [1/x]$. In four dimensions, we have $\int d^4p\delta^{(4)}(p) = 1$, such that $[\delta^{(4)}(p)] = [1/p^4] = \text{GeV}^{-4}$.

Each factor of $d^3\vec{p}$ has units of energy cubed, $[d^3\vec{p}] = \text{GeV}^3$. Each factor of $1/2E$ has units of inverse energy, $[1/2E] = \text{GeV}^{-1}$. Putting this all together, we find that the phase space measure has units

$$[d\Pi_f] = \text{GeV}^{-4}(\text{GeV}^3\text{GeV}^{-1})^N = \text{GeV}^{2N-4}.$$

Therefore, for the units to match in $[\Gamma] = \text{GeV}$, we need

$$[\mathcal{M}]^2 = \frac{[\Gamma][V]^{M-1}[E]^M}{[d\Pi_f]} = \frac{\text{GeV}\,\text{GeV}^{-3(M-1)}\,\text{GeV}^M}{\text{GeV}^{2N-4}},$$

or

$$[\mathcal{M}] = \text{GeV}^{4-N-M}.$$

For example, for a two-body decay $1 \to 2$ particles, we have $N = 2$ and $M = 1$, such that $[\mathcal{M}] = \text{GeV}$. For a $2 \to 2$ scattering process, we have $N = 2$ and $M = 2$, such that $[\mathcal{M}] = \text{GeV}^0$, i.e., dimensionless.

In scattering processes, it is more convenient to express the rate in terms of a cross section $\sigma$, which has units of area, $[\sigma] = \text{GeV}^{-2}$. You can convince yourself of this by noting that the rate of scattering events per unit time is given by $\Gamma = \sigma \times \text{flux}$, where the flux has units of energy cubed, $[\text{flux}] = \text{GeV}^3$. Therefore, $[\sigma] = [\Gamma/\text{flux}] = \text{GeV}/\text{GeV}^3 = \text{GeV}^{-2}$. We will derive the expression for the cross section below.

**Decay $1 \to N$ particles.** Let us simplify this expression for the case of a one-particle decay, where $M = 1$ and

$$P \to p_1 + p_2 + \cdots + p_N.$$

In this case, the volume factor disappears completely, and we can go to the rest frame of the decaying particle, where $P = (m_P, \vec{0})$, to write

$$\Gamma = \int \frac{|\mathcal{M}_{i' \to f'}|^2}{2E_P} (2\pi)^4 \delta^{(4)}(p_i - p_f) \prod_{j=1}^{N} \frac{d^3p_j}{(2\pi)^3 2E_j} = \frac{1}{2m_P} \int d\Pi_f |\mathcal{M}|^2. \tag{1.17}$$

where $d\Pi_f$ is the Lorentz-invariant phase space measure for the final state particles, each with 4-momentum $p_j$, $\mathcal{M}$ is the invariant matrix element (amplitude) for the process, and the delta function enforces energy-momentum conservation.

**Two-body decays.** Let us investigate the simplest non-trivial case: the two-body decay $P \to p_1 + p_2$. The phase space measure can be simplified as follows

$$\begin{aligned}\int d\Pi_f &= \int (2\pi)^4 \delta^{(4)}(P - p_1 - p_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \\ &= \int \delta^{(0)}(E_P - E_1 - E_2)|\vec{p}_1|^2 d|\vec{p}_1| d\Omega_1 \left(\frac{1}{16\pi^2 E_1 E_2}\right) \\ &= \int \frac{d\Omega_1}{4\pi} \left(\frac{|\vec{p}_1|}{4\pi E_2}\right)\end{aligned}$$

where we used the fact that $d|\vec{p}| = \frac{E}{|\vec{p}|} dE$ to integrate over the last delta function.

Only after integrating over the delta function we can enforce energy-momentum conservation, $\vec{P} = \vec{p}_1 + \vec{p}_2$ as well as $E_P = E_1 + E_2$. In the rest frame of the decaying particle, $\vec{P} = 0$, so $\vec{p}_1 = -\vec{p}_2$ and both final states have the same momentum

$$|\vec{p}| = |\vec{p}_1| = |\vec{p}_2|.$$

In the massless case, $E_i = |\vec{p}_i|$, and you get $|\vec{p}| = E_1 = E_2 = m_P/2$. Putting this all together, we get

$$\int d\Pi_f = \frac{1}{4\pi} \int \frac{d\Omega}{4\pi} \quad \text{and} \quad \Gamma = \frac{1}{2m_P} \frac{\int \frac{d\Omega}{4\pi} |\mathcal{M}|^2}{4\pi}.$$

Clearly, for units to match $[\Gamma] = \text{GeV}$, we need $[\mathcal{M}] = \text{GeV}$.

So, for a two-body decay, the decay rate comes with a factor of $1/4\pi$ phase-space factor, as well as the normalization factor $1/2m_P$.

Show that for a two-body decay $P \to p_1 + p_2$ where the final states have masses $m_1$ and $m_2$, the momentum of each particle in the rest frame of $P$ is given by

$$|\vec{p}| = \frac{\sqrt{(m_P^2 - (m_1 + m_2)^2)(m_P^2 - (m_1 - m_2)^2)}}{2m_P}.$$

**Three-body decays.** Let us see how the phase space factor changes things for a three-body decay $P \to p_1 + p_2 + p_3$. In this case, we have

$$
\begin{aligned}
\int d\Pi_f &= \int \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^{(4)}(P - p_1 - p_2 - p_3) \qquad (1.18)\\
&= \int d|\vec{p}_1|d\Omega_1 d|\vec{p}_2|d\Omega_2 \left(\frac{\delta^{(0)}(E_P - E_1 - E_2 - E_3)|\vec{p}_1|^2|\vec{p}_2|^2}{256\pi^5 E_1E_2E_3}\right)\\
&= \left(\frac{1}{16\pi^3}\right)\int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}dE_1dE_2\left(\frac{|\vec{p}_1||\vec{p}_2|}{E_3}\right)\delta^{(0)}(E_P - E_1 - E_2 - E_3)
\end{aligned}
$$

In the massless limit, $E_i = |\vec{p}_i|$, and we can do the integral over $E_2$ using the delta function to get

$$\int d\Pi_f = \left(\frac{1}{4\pi}\frac{1}{16\pi^2}\right)\int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}dE_1\left(\frac{4E_1E_2}{m_P - E_1 - E_2}\right) \qquad (1.19)$$

where we used $E_3 = m_P - E_1 - E_2$. We are not actually going to do the integral just yet, but we are starting to see how the phase space factors start to add up as you add more particles. In practice, there are way more convenient parameterizations of the three-body phase space that make it easier to do the integrals. We will outline them later.

**Scattering $2 \to N$ particles.** For a scattering process with two particles in the initial state, $M = 2$, we have

$$d\Gamma = \frac{|\mathcal{M}_{i'\to f'}|^2}{V}(2\pi)^4\delta^{(4)}(p_i - p_f)\frac{1}{2E_1 2E_2}\prod_{\ell=1}^{N}\frac{d^3p_\ell}{(2\pi)^3 2E_\ell}. \qquad (1.20)$$

This is often rewritten in terms of the cross section $\sigma$, which is defined as the rate per flux of incoming particles. The flux is defined as the number of incoming particles per unit area per unit time. For two incoming particles with 4-momenta $p_1$ and $p_2$, the flux is given by

$$\text{Flux} = \frac{v_{\text{rel}}}{E_1E_2V}. \qquad (1.21)$$

This form comes from the fact that the flux should be proportional to the relative velocity between the two incoming particles, which is given by $v_{\text{rel}} = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}/(E_1E_2)$, as well as the density of incoming particles, which is $1/V$ for a single particle in a box of volume $V$. You can see this by noting that $\vec{v} = \vec{p}/E$, such that:

$$p_1 = (E_1, E_1\vec{v}_1), \quad p_2 = (E_2, E_2\vec{v}_2) \implies p_1\cdot p_2 = E_1E_2(1 - \vec{v}_1\cdot\vec{v}_2),$$

The cross section is then defined as

$$\boxed{\sigma = \frac{d\Gamma}{\text{Flux}} = \frac{|\mathcal{M}_{i'\to f'}|^2}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}(2\pi)^4\delta^{(4)}(p_i - p_f)\prod_{\ell=1}^{N}\frac{d^3p_\ell}{(2\pi)^3 2E_\ell}.} \qquad (1.22)$$

Note how this is explicitly Lorentz-invariant, as it should be.

## 2 Decomposition into two-body phase spaces

We now derive a key result for the phase space treatment in calculating cross sections and decay rates. We begin with the factorization of $N$-final state phase-space factors into $N-2$ two-body ones. In general, the $N$-body phase space can be written as

$$d\Pi_N(P,\{p_i\}) = (2\pi)^4\delta^4(P-\sum_i^N p_i)\prod_i^N \frac{\mathrm{d}^3p_i}{(2\pi)^3 2E_i}, \tag{2.1}$$

where $\{p_i\} = p_1,\ldots,p_N$. Focusing on the 1-2 subsystem with total momentum $p_{12} = p_1+p_2$, we can write

$$\begin{aligned} d\Pi_N(P,\{p_i\}) &= \int \mathrm{d}^4p_{12}\,\delta^4(p_{12}-p_1-p_2)\,(2\pi)^4\delta^4(P-\sum_i^N p_i)\prod_i^N \frac{\mathrm{d}^3p_i}{(2\pi)^3 2E_i} \\ &= \int \mathrm{d}^4p_{12}\,d\Pi_2(p_{12},p_1,p_2)\,(2\pi)^4\delta^4(P-p_{12}-\sum_{i=3}^N p_i)\prod_{i=3}^N \frac{\mathrm{d}^3p_i}{(2\pi)^3 2E_i} \\ &= \int \mathrm{d}^4p_{12}\,\mathrm{d}m_{12}^2\,\delta(p_{12}^2-m_{12}^2)\,d\Pi_2(p_{12},p_1,p_2) \\ &\qquad\qquad \times (2\pi)^4\delta^4(P-p_{12}-\sum_{i=3}^N p_i)\prod_{i=3}^N \frac{\mathrm{d}^3p_i}{(2\pi)^3 2E_i}, \end{aligned} \tag{2.2}$$

which from

$$\int \mathrm{d}^4p_{12}\,\delta(p_{12}^2-m_{12}^2) = \int \mathrm{d}^4p_{12}\,\frac{\delta(E_{12}-\sqrt{m_{12}^2+|\vec{p}_{12}|^2})}{2\sqrt{m_{12}^2+|\vec{p}_{12}|^2}} = \int \frac{\mathrm{d}^3p_{12}}{2E_{12}},$$

yields the final results

$$d\Pi_N(P,p_1,\ldots,p_N) = \frac{\mathrm{d}m_{12}^2}{2\pi} d\Pi_2(p_{12},p_1,p_2) d\Pi_2(P,p_{12},p_3,\ldots,p_N). \tag{2.3}$$

This result not only lets us factorize any resonant features in phase space, but also provides a recipe to tackle the kinematics of any process in terms of a series of 2-body problems, which are much simpler. The 2-body phase-space factors and associated four-momenta in the respective center-of-mass (CM) frame can always be written as

$$\begin{aligned} d\Pi_2(p_{12},p_1,p_2) &= \frac{\lambda^{1/2}\left(1, m_1^2/E_{12}^{\mathrm{CM}\,2}, m_2^2/E_{12}^{\mathrm{CM}\,2}\right)}{32\pi^2}\mathrm{d}\Omega^{\mathrm{CM}}, \\ p_{12} &= \left(E_{12}^{\mathrm{CM}}, \vec{0}\right), \\ p_1 &= \left(\frac{E_{12}^{\mathrm{CM}\,2}+m_1^2-m_2^2}{2E_{12}^{\mathrm{CM}\,2}}, |\vec{p}_1|\sin\theta\cos\phi, |\vec{p}_1|\sin\theta\sin\phi, |\vec{p}_1|\cos\theta\right), \\ p_2 &= \left(\frac{E_{12}^{\mathrm{CM}\,2}+m_2^2-m_1^2}{2E_{12}^{\mathrm{CM}\,2}}, -\vec{p}_1\right), \end{aligned} \tag{2.4}$$

where $\lambda(a,b,c) = (a-b-c)^2 - 4bc$ is the Källén function. Now, the problem is reduced to finding the CM frame of every $p_{ij}$ subsystem, and the transformation between all such frames, if necessary. Of course, the dependence of the matrix element on the kinematics makes certain phase space parametrization better than others, making each problem unique.

## 3 Some Three-body decay examples

**Three-body decays (Dalitz variables)** Here we exploit the phase space factorization

$$d\Pi_3(P;p_1,p_2,p_3) = \frac{\mathrm{d}m_{12}^2}{2\pi} d\Pi_2(P;p_{12},p_3) d\Pi_2(p_{12};p_1,p_2), \tag{3.1}$$

where $p_{12} = p_1 + p_2$ and we define the three-body decay Dalitz variables

$$\begin{aligned} m_{12}^2 &= (P-p_3)^2 = M^2 + m_3^2 - 2ME_3 \\ m_{13}^2 &= (P-p_2)^2 = M^2 + m_2^2 - 2ME_2 \\ m_{23}^2 &= (P-p_1)^2 = (p_2+p_3)^2 = m_2^2 + m_3^2 + 2E_2E_3(1-\beta_2\beta_3\cos\theta_{23}), \end{aligned} \tag{3.2}$$

where $\beta_i = |\vec{p_i}|/E_i$ and $m_{12}^2 + m_{13}^2 + m_{23}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$. Now, we can always write the two-body phase space factors as

$$d\Pi_2(p_{12};p_1,p_2) = \frac{\mathrm{d}^3\vec{p_1}}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\vec{p_2}}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}\left(p_{12} - p_1 - p_2\right) \tag{3.3}$$

$$d\Pi_2(P;p_{12},p_3) = \frac{\mathrm{d}^3\vec{p_{12}}}{(2\pi)^3 2E_{12}} \frac{\mathrm{d}^3\vec{p_3}}{(2\pi)^3 2E_3} (2\pi)^4 \delta^{(4)}\left(P - p_{12} - p_3\right), \tag{3.4}$$

but it is only in the rest frame of $p_{12}$ and $P$, respectively, that we can use the simplifications

$$d\Pi_2(p_{12};p_1,p_2) = \lambda^{1/2}(1, m_1^2/m_{12}^2, m_2^2/m_{12}^2) \frac{\mathrm{d}\Omega_{12}}{32\pi^2}, \tag{3.5}$$

$$d\Pi_2(P;p_{12},p_3) = \lambda^{1/2}(1, m_{12}^2/M^2, m_3^2/M^2) \frac{\mathrm{d}\Omega_3}{32\pi^2}, \tag{3.6}$$

where $\lambda(a,b,c) = (a-b-c)^2 - 4bc$ is the Källen function, and $\Omega_{12}$ is the solid angle of $\vec{p_2}$ with respect to the vector $\vec{p_1}$ in the rest frame of $p_{12}$. While it is possible to proceed this way and find the transformation between the two frame, it is much more convenient to work in a single reference frame. Picking the rest frame of $P$, we can then parametrize the general three-body phase space using Eqs. (3.3) and (3.6). Let us simplify these expressions further.

First, we integrate over $\vec{p_2}$ and $\cos\theta_{12}$ to eliminate the delta function. After that, momentum and energy conservation in the full system is guaranteed, and we end up with the following useful relations

$$\cos\theta_{13} = \frac{2E_1E_3 + m_{12}^2 + m_{23}^2 - m_2^2 - M^2}{2|\vec{p_1}||\vec{p_3}|}, \tag{3.7}$$

$$E_1 = \frac{M^2 + m_1^2 - m_{23}^2}{2M}, \qquad |\vec{p_1}| = \lambda^{1/2}\left(1, m_{23}^2/M^2, m_1^2/M^2\right)\frac{M}{2} \tag{3.8}$$

$$E_3 = \frac{M^2 + m_3^2 - m_{12}^2}{2M}, \qquad |\vec{p_3}| = \lambda^{1/2}\left(1, m_{12}^2/M^2, m_3^2/M^2\right)\frac{M}{2} \tag{3.9}$$

$$E_2 = \frac{m_{23}^2 + m_{12}^2 - m_3^2 - m_1^2}{2M}, \qquad |\vec{p_2}| = \lambda^{1/2}\left(1, m_{13}^2/M^2, m_2^2/M^2\right)\frac{M}{2}. \tag{3.10}$$

Now, the final form for the three-body phase space is

$$d\Pi_3(P; p_1, p_2, p_3) = \frac{\mathrm{d}|\vec{p_1}|\,\mathrm{d}|\vec{p_3}|\,\mathrm{d}\Omega_3\,\mathrm{d}\phi_{13}}{8(2\pi)^5\beta_1\beta_3} \tag{3.11}$$

$$= \frac{\mathrm{d}m_{12}^2\,\mathrm{d}m_{23}^2\,\mathrm{d}\Omega_3\,\mathrm{d}\phi_{13}}{32M^2(2\pi)^5}. \tag{3.12}$$

Now, we only need to find the limits for the Dalitz variables. Already from Eq. (3.1) in the rest frame of $p_1$ we can deduce the limits of $m_{12}^2$. Recall that all configurations are possible, so we may find the range of values for $m_{12}^2$ by considering two limits: *i)* $m_{12}^2$ is so large that both $p_3$ and $p_{12}$ are at rest ($E_3 = m_3$, $E_{12} = m_{12}$), and *ii)* $m_{12}^2$ is so small that $p_1$ and $p_2$ are at rest in the rest frame of $p_{12}$ ($E_1 = m_1$, $E_2 = m_2$). This is equivalent to

$$(m_1 + m_2)^2 \le m_{12}^2 \le (M - m_3)^2. \tag{3.13}$$

Now that the absolute limits of $m_{12}^2$ are specified, we can derive the $m_{23}^2$ limits. This is most easily done in the rest frame of $p_{12}$, where we shall denote our variables with an asterisk $^*$. Taking $p_{12}^* = (m_{12}, \vec{0})$, we can easily find

$$E_P^* = \frac{m_{12}^2 + M^2 - m_3^2}{2m_{12}}, \quad E_3^* = -\frac{m_{12}^2 - M^2 + m_3^2}{2m_{12}} \tag{3.14}$$

$$E_2^* = \frac{m_{12}^2 + m_2^2 - m_1^2}{2m_{12}}, \quad E_1^* = \frac{m_{12}^2 - m_2^2 + m_1^2}{2m_{12}} \tag{3.15}$$

noting the minus sign for $E_P^*$. Now, it is trivial to find the limits by taking $c_{P1}^* = \pm 1$:

$$(m_{23}^2)_{\text{min}}^{\text{max}} = M^2 + m_1^2 - 2E_P^*E_1^* \pm 2|\vec{P}|^*|\vec{p_1}|^*. \tag{3.16}$$

Note that $|\vec{P}|^* = |\vec{p_3}|^*$, $|\vec{p_1}|^* = |\vec{p_2}|^*$, and $c_{P1}^* = -c_{23}^*$, so that we can also state

$$(m_{23}^2)_{\text{min}}^{\text{max}} = m_2^2 + m_3^2 + 2E_2^*E_3^* \pm 2|\vec{p_2}|^*|\vec{p_3}|^*. \tag{3.17}$$

Our phase space factor and limits agree with those shown in the PDG review of kinematics.

**Total phase-space volume.** Now we can check that in the massless limit, we reproduce the well-known result

$$\int d\Pi_3(P; p_1, p_2, p_3) = \frac{M^2}{32(2\pi)^5}\int_0^1\int_0^{1-x_{12}} \mathrm{d}x_{12}^2\,\mathrm{d}x_{13}^2, \tag{3.18}$$

$$= \frac{M^2}{256\pi^3}. \tag{3.19}$$

**Three-body decays (fractional energies)** We now revisit the phase space parameterization above with another popular scheme based on fractional energies for the daughter particles using the following invariants,

$$x \equiv \frac{2(P\cdot k_3)}{M^2} = \frac{2E_3^{\rm CM}}{M}, \quad y \equiv \frac{2(P\cdot k_1)}{M^2} = \frac{2E_1^{\rm CM}}{M}. \tag{3.20}$$

We can further define,

$$z \equiv \frac{2(P\cdot k_2)}{M^2} = \frac{2E_2^{\rm CM}}{M} = 2 - x - y, \quad r_i = \frac{m_i}{M}, \quad x_{ij} \equiv \frac{m_{ij}^2}{M^2} \tag{3.21}$$

In terms of the fractional Dalitz variables,

$$x_{12} = 1 + r_3^2 - x, \quad x_{23} = 1 + r_1^2 - y. \tag{3.22}$$

In terms of Eq. (3.20), we rewrite Eq. (3.12),

$$d\Pi_3(P; p_1, p_2, p_3) = M^2 \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\Omega_3\,\mathrm{d}\phi_{13}}{32(2\pi)^5}. \tag{3.23}$$

Now onto the limits on $x$ and $y$. From Eq. (3.13) and Eq. (3.16), we deduce,

$$2r_3 < x < 1 + r_3^2 - (r_1 + r_2)^2, \tag{3.24}$$

$$1 + r_1^2 - x_{23}^{\rm max} < y < 1 + r_1^2 - x_{23}^{\rm min}. \tag{3.25}$$

In the special case where $r_2 = r_3 = 0$ (as in $\pi^+ \to e^+\nu_e\gamma$), and inverting the order of integration, we recover the well-known expressions,

$$2r_1 < y < 1 + r_1^2, \tag{3.26}$$

$$x_{\rm min}^{\rm max} = 1 - \frac{y}{2} \pm \frac{1}{2}\sqrt{y^2 - r_1^2}. \tag{3.27}$$