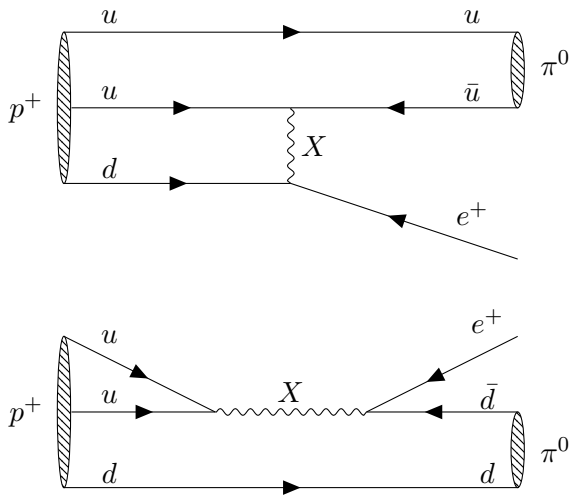
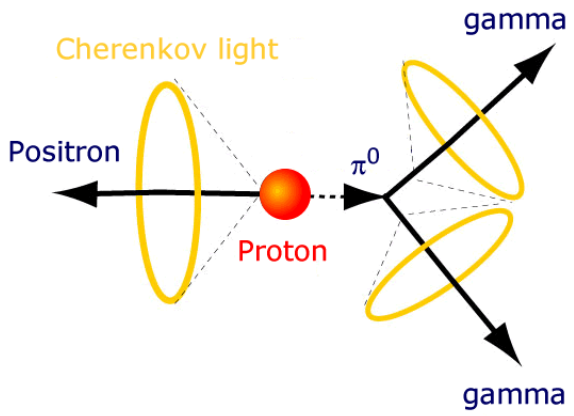
Solutions and discussion for Assignment 2. This question is inspired by the fact that many GUTs predict proton decay, and the current experimental limits on the proton lifetime already rule out some of these models. The current best limit on the proton lifetime is about $\tau_p > 2.4 \times 10^{34}$ years from the Super-Kamiokande experiment (https://inspirehep.net/literature/1827343). The future Hyper-Kamiokande experiment is expected to improve this limit by about an order of magnitude.

Question 1. A π^0 is a neutral meson composed of a quark-antiquark pair, specifically a combination of up and down quarks: $\pi^0 \sim \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$. To get a π^0 then, all we need to do is to create a quark-antiquark pair of either $u\bar{u}$ or $d\bar{d}$. There are many possibilities for the Feynman diagram, including t-channel and the s-channel diagrams. You could also have used some gluons to create quark-antiquark pairs.



In the detectors, these proton decay events look extraordinarily clean, as the positron and the two photons from the $\pi^0 \to \gamma\gamma$ decay all originate from a single point in the detector.



Question 2. The process is $p^+ \to e^+ + \pi^0$. On the LHS, we have $B(p^+) = +1$ and $L(p^+) = 0$, while on the RHS, we have $B(e^+) = 0$, $L(e^+) = -1$, $B(\pi^0) = 0$, and $L(\pi^0) = 0$. So, we have $\Delta B = -1$ and $\Delta L = -1$, but $\Delta(B - L) = 0$. Thus, the combination $B - L$ remains conserved in this process.

It turns out that $B - L$ holds a special place within the Standard Model. If you recall, we argued that B and L turn out to individually be conserved in the Standard Model, accidentally. However, one can show that B and L are actually violated by non-perturbative phenomena in the Standard Model. So while they appear to be conserved in every vertex of Feynman diagrams, there are processes where perturbation theory cannot be applied to which do violate them. Therefore, they do not constitute true underlying symmetries of the Standard Model and are said to be *anomalous*.[8] These non-perturbative effects are extremely small at low energies, so for all practical purposes, B and L appear to be conserved in our everyday world, though that may not be the case for processes at extremely high temperatures (such as in the early universe) or high energies. The combination $B - L$, on the other hand, is indeed conserved at all orders of perturbation theory and is said to be "anomaly-free". It can, then, be a true symmetry of the theory.

This is also important for neutrinos. As we saw, neutrino masses require new physics beyond the Standard Model. If neutrinos are Majorana particles, then lepton number is violated by two units. Because neutrino carry no baryon number, their Majorana masses would also violate $B - L$ by two units.

Question 3. What I am really looking for here is

$$\Gamma_p \sim \left(\frac{g_X^2}{m_X^2}\right)^2 \times m_p^5 \times \frac{1}{\mathcal{O}(4\pi)},$$

where the first factor comes from the two vertices in the Feynman diagram and the heavy propagator of X (after squaring the amplitude), the second factor comes from dimensional analysis (the only energy scale in the problem is $m_p = 938$ GeV $> m_{\pi^0}$), and the last one from the consideration that this is a 2-body decay. This is a consequence of the fact that the X boson is extremely heavy, so the propagator factor is approximately just a constant $1/m_X^2$, which gets squared in the rate. The proton lifetime is then

$$\tau_p = \frac{1}{\Gamma_p} \sim \frac{m_X^4}{g_X^4\, m_p^5}.$$

Recall: for 2-body processes, we start with a factor of 4π in the denominator and add an overall $\frac{1}{16\pi^2}$ for each additional particle in the final state (or for each loop in the Feynman diagram, it turns out). Please see the phase-space notes for more details on this naive dimensional analysis.

[8] Keywords here are: anomalies in QFT, triangle diagrams, sphalerons, etc.

Question 4. The number of free protons in Hyper-Kamiokande is just

$$N_p = \frac{M_{\rm det}}{m_p} \times \frac{2}{2+16}$$

where $M_{\rm det}$ is the total mass of the water in the detector, m_p is the mass of the proton (approximately the same as neutrons and neglecting the electron mass and binding energies). We also used the fact that there are 2 free protons per water molecule, H_2O. The volume of the detector is approximately 0.4 Mton $= 4 \times 10^{11}$ g. Using $m_p \simeq 1.67 \times 10^{-24}$ g, we find

$$N_p \simeq \frac{4 \times 10^{11}\ {\rm g}}{1.67 \times 10^{-24}\ {\rm g}} \times \frac{2}{18} \simeq 2.7 \times 10^{34}.$$

If we do not observe any proton decay events in 10 years of operation, then

$$N_{\rm decays} < 1 \implies N_p(1 - e^{-t/\tau_p}) \simeq N_p \frac{t}{\tau_p} < 1 \tag{3.49}$$

with $t = 10$ years. In other words, we can set a limit on the proton lifetime as

$$\tau_p > N_p \times t = 2.7 \times 10^{34} \times 10 \text{ years} \simeq 2.7 \times 10^{35} \text{ years}.$$

This is about 2×10^{25} times the age of the universe. You could have been fancier here and derived a 90% C.L. limit using Poisson statistics, for which the requirement is stronger, $N_{\rm decays} < 2.3$.

Question 5. This depends on what you took for the $\mathcal{O}(1)$ factors in your decay rate. Using the expression from above with 4π in the denominator:

$$\tau_p > 2.7 \times 10^{35} \text{ years} \implies m_X > (g_X) \times (m_p^5 \times 2.7 \times 10^{35} \text{ years}/4\pi)^{1/4}. \tag{3.50}$$

From $\hbar \simeq 6.6 \times 10^{-16}$ eV . s and year $= \pi \times 10^7$ s, we can find 1 year $\simeq 4.7 \times 10^{31}\ {\rm GeV}^{-1}$. Using $m_p \simeq 0.938$ GeV and setting $g_X \simeq 1$, we find

$$m_X > (0.938^5 \times 2.7 \times 10^{35} \times 4.7 \times 10^{31}/4\pi)^{1/4} \text{ GeV} \simeq 2.9 \times 10^{16} \text{ GeV}.$$

The scale of new physics is therefore constrained to be *above* this value. Alternatively, you could also say that

$$\frac{m_X}{g_X} > 2.9 \times 10^{16} \text{ GeV}, \tag{3.51}$$

where the mass reach is weaker for smaller couplings. This is an extremely high energy scale, about three orders of magnitude above the reach of the LHC, and about 3 orders of magnitude below the Planck scale.

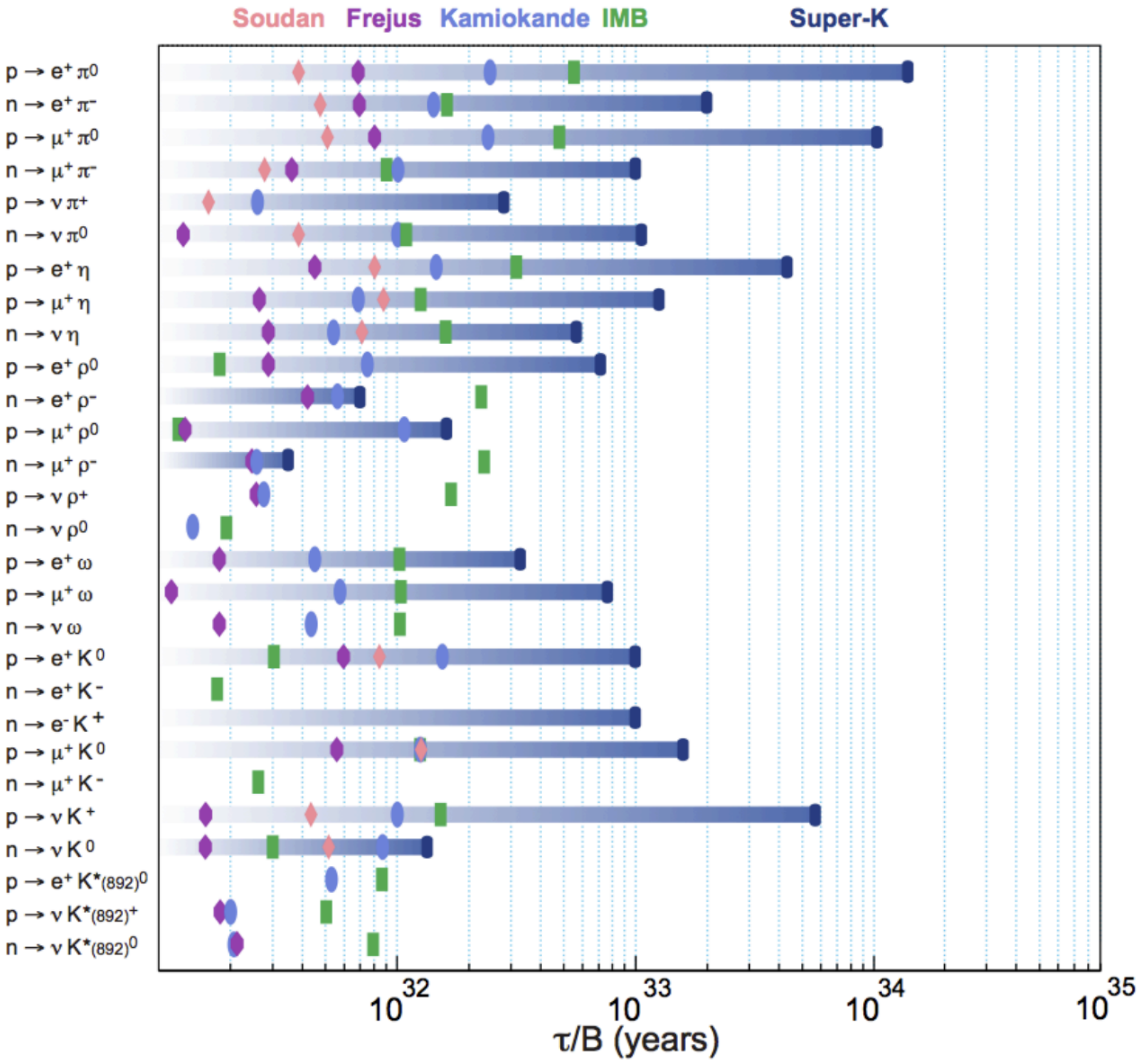
Question 6. We are a factor of $\gtrsim 300$ away from the Planck scale. Since the proton lifetime scale as the fourth power of the mediator mass m_X, so we need to run Hyper-Kamiokande $(300)^4 \simeq 10^{10}$ times longer, which is about $10^{10} \times 10$ years or about 10 times the age of the universe! Your answer may vary depending on the $\mathcal{O}(1)$ factors you used. Note that even a factor of 2 here gets exponentiated by 4, so the answer can vary by one or

two orders of magnitude. Similarly, if your coupling was a bit smaller than 1, that would also weaken the limit on m_X and increase the required time by $1/g_X^4$.

Where to go from here? Well, we could try to build a larger detector. Let us imagine instead that you can use the largest neutrino detector ever conceived: TRIDENT in the south China sea, a factor of several larger volume than IceCube, in the South Pole. These detectors cannot resolve the proton decay signature nearly as well as Hyper-Kamiokande, but let us imagine you could for now so that every proton decay counts as a smoking-gun signal. TRIDENT wants to instrument a volume of about 7 km^3 or 7 Gton of water (about 10^4 times larger than Hyper-Kamiokande). They can do this because they are not building a tank, they are just putting strings of photomultiplier tubes in the ocean and using the water as the detector volume (just like IceCube does in the ice). If we operate it for an entire human career, say 50 years (if you are lucky), we have an exposure (mass $\times$ time) that is about 5×10^4 larger than that of Hyper-Kamiokande would have. In that case, we reach $m_X \sim 4.3 \times 10^{17}$ GeV, which is better, but all this only bought us another order of magnitude in m_X.

So, in summary, while proton decay gives us a unique window into physics at extremely high energies, it is very challenging to improve on the current limits. Progress in terms of the scale of new physics is incremental at best, so having probed up to about 10^{16} GeV, we are still unlikely to reach scales like the Planck scale anytime soon.

Let me also end by saying that this channel is not the only one that can be used to search for the violation of baryon number. Other channels include $p^+ \to K^+ + \bar{\nu}$, $p^+ \to \mu^+ + \pi^0$, $p^+ \to e^+ + \gamma$, etc. Channels with neutrinos in the final state are harder because neutrinos fly right out of your detector without interacting, so you have to rely on the kinematics of the visible particles to determine if they came from proton decay or other background processes (we're dealing with some extremely rare events). Neutron decay is another example. Free neutrons decay, but neutrons bound in nuclei are stable(!) If a decay channel like $n \to e^- + \pi^+$ were observed, that would also be a sign of baryon number violation. In this case you have to worry about the nuclear physics of the neutron bound in the nucleus, which makes things more complicated, but it is still a possible probe. Other examples include neutron-antineutron oscillations ($n \leftrightarrow \bar{n}$), dinucleon decays ($nn \to \pi^+\pi^-$ or $nn \to e^+e^-$), etc. Below, you can find a figure that shows the current limits on different proton and neutron decay channels.



(Current limits on nucleon decays for different decay channels. The different colored markers indicate different experiments.)