Solutions and discussion for Assignment 3.

Question 1. Let us start with the time-evolution of a neutrino state in the mass basis. In the presence of a non-Hermitian term in the Hamiltonian, we have

$$|\nu_i(t)\rangle = e^{-i\hat{H}_0 t - \frac{1}{2}\hat{\Gamma}t}\,|\nu_i\rangle = e^{-iE_i t - \frac{1}{2}\Gamma_i t}\,|\nu_i\rangle\,, \tag{1.4}$$

where we expanded the exponential and used that $\hat{H}_0\,|\nu_i\rangle = E_i\,|\nu_i\rangle$ and $\hat{\Gamma}\,|\nu_i\rangle = \Gamma_i\,|\nu_i\rangle$.

The oscillation probability is then given by

$$P(\nu_\alpha \to \nu_\beta) = |\,\langle\nu_\beta|\nu_\alpha(t)\rangle\,|^2 = |\,\langle\nu_\beta|\sum_i e^{-i\hat{H}_0 t - \frac{1}{2}\hat{\Gamma}t} U^*_{\alpha i}\,|\nu_i\rangle\,|^2 \tag{1.5}$$

$$= \left|\langle\nu_\beta|\sum_i e^{-iE_i t - \frac{1}{2}\Gamma_i t} U^*_{\alpha i}\,|\nu_i\rangle\right|^2 \tag{1.6}$$

$$= \left|\sum_{i,j} U^*_{\alpha i} U_{\beta j} e^{-iE_i t - \frac{1}{2}\Gamma_i t}\,\langle\nu_j|\nu_i\rangle\right|^2 \tag{1.7}$$

$$= \left|\sum_i U^*_{\alpha i} U_{\beta i} e^{-iE_i t - \frac{1}{2}\Gamma_i t}\right|^2 \tag{1.8}$$

$$= \left(\sum_j U^*_{\alpha j} U_{\beta j} e^{-iE_j t - \frac{1}{2}\Gamma_j t}\right)^* \left(\sum_i U^*_{\alpha i} U_{\beta i} e^{-iE_i t - \frac{1}{2}\Gamma_i t}\right) \tag{1.9}$$

$$\tag{1.10}$$

$$= \sum_{i,j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} e^{-i(E_i - E_j)t} e^{-\frac{1}{2}(\Gamma_i + \Gamma_j)t} \tag{1.11}$$

$$= \sum_{i,j} A^{ij}_{\alpha\beta} e^{-i(E_i - E_j)t} e^{-\frac{1}{2}(\Gamma_i + \Gamma_j)t}, \tag{1.12}$$

where we used $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}\,|\nu_i\rangle$ and $|\nu_\beta\rangle = \sum_j U^*_{\beta j}\,|\nu_j\rangle$ as well as the orthonormality of the mass eigenstates, $\langle\nu_j|\nu_i\rangle = \delta_{ij}$. Using the same momentum approximation, $E_i \approx p + \frac{m_i^2}{2p}$, we have $E_i - E_j \approx \frac{\Delta m^2_{ij}}{2p}$. Keeping the leading terms Δm^2, we use $p \simeq E_\nu$ and $t \approx L$ to write

$$P(\nu_\alpha \to \nu_\beta) = \sum_{i,j} A^{ij}_{\alpha\beta} e^{-i\Delta_{ij}} e^{-\gamma_{ij}}, \quad \text{with} \quad \Delta_{ij} = \frac{\Delta m^2_{ij} L}{2E_\nu}, \quad \gamma_{ij} = \frac{(\Gamma_i + \Gamma_j)t}{2}. \tag{1.13}$$

We can split the sum into diagonal ($i = j$) and off-diagonal ($i \neq j$) terms:

$$P(\nu_\alpha \to \nu_\beta) = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 e^{-\Gamma_i t} + \sum_{i \neq j} A^{ij}_{\alpha\beta} e^{-i\Delta_{ij}} e^{-\gamma_{ij}}. \tag{1.14}$$

The off-diagonal sum can be simplified by noting that the terms with $i < j$ are complex conjugates of the terms with $i > j$. Therefore,

$$\sum_{i \neq j} U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j} e^{-i\Delta_{ij}} e^{-\gamma_{ij}} = 2 \sum_{i > j} \mathrm{Re}\left(U_{\beta i} U^*_{\alpha i} U^*_{\beta j} U_{\alpha j} e^{-i\Delta_{ij}}\right) e^{-\gamma_{ij}}. \tag{1.15}$$

Note that

$$\mathrm{Re}\left(U_{\beta i}U^*_{\alpha i}U^*_{\beta j}U_{\alpha j}e^{-i\Delta_{ij}}\right) = \mathrm{Re}(A^{ij}_{\alpha\beta})\cos\Delta_{ij} + \mathrm{Im}(A^{ij}_{\alpha\beta})\sin\Delta_{ij}, \tag{1.16}$$

so that we finally have

$$P(\nu_\alpha \to \nu_\beta) = \sum_i |U_{\alpha i}|^2|U_{\beta i}|^2 e^{-\Gamma_i t} + 2\sum_{i>j} e^{-\gamma_{ij}}\left[\mathrm{Re}(A^{ij}_{\alpha\beta})\cos\Delta_{ij} + \mathrm{Im}(A^{ij}_{\alpha\beta})\sin\Delta_{ij}\right], \tag{1.17}$$

which proves the proposed result.

Question 2. In the limit $\Gamma_i \to 0$, we recover the standard oscillation probability in vacuum, Eq. (1.1), as expected. To see that, note that $\delta_{\alpha\beta} = \sum_i |U_{\alpha i}|^2|U_{\beta i}|^2 + \sum_{i>j} 2\,\mathrm{Re}(A^{ij}_{\alpha\beta})$.

In the limit $\Gamma_i t \gg 1$, all terms are exponentially suppressed, and for any α and β, we have $P(\nu_\alpha \to \nu_\beta) \to 0$. This means that neutrinos are leaving our system, and cannot be detected anymore. One interpretation for this is that neutrinos are decaying into states that we are not explicitly tracking (say, sterile neutrinos or other particles beyond the Standard Model), which leads to a loss of the neutrino wavefunction.

In intermediate regimes, we can have a partial loss of neutrinos, which can lead to a damping of the oscillation pattern.

Question 3. The ν_e survival probability is given by

$$P(\nu_e \to \nu_e) = \sum_i |U_{ei}|^4 e^{-\Gamma_i t} + 2\sum_{i>j} |U_{ei}|^2|U_{ej}|^2 e^{-\gamma_{ij}}\cos\Delta_{ij}, \tag{1.18}$$

where we used the fact that $A^{ij}_{ee} = |U_{ei}|^2|U_{ej}|^2$ is real. In the limit $\Delta_{31} \gg 1$, the oscillations driven by Δ_{31} and Δ_{32} have averaged out, so that $\langle\cos\Delta_{31}\rangle = \langle\cos\Delta_{32}\rangle = 0$.

Let us prove this explicitly. Firstly, in the limit $\Delta_{31} \gg 1$, the oscillations due to Δ_{31} are very fast, so that we can average over one period:

$$\langle\cos\Delta_{31}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\cos\Delta_{31}d\Delta_{31} = 0. \tag{1.19}$$

One can be a little more precise and average over the observable $x = L/E_\nu$ instead since those are the quantities that vary in our experiments. In that case, we average over an interval $\delta x = x_1 - x_0$, so that

$$\langle\cos\Delta_{31}\rangle = \frac{1}{\delta x}\int_{x_0}^{x_1}\cos\left(\frac{\Delta m^2_{31}}{2}x\right)dx = \frac{\sin\left(\frac{\Delta m^2_{31}}{2}(x_0+\delta x)\right) - \sin\left(\frac{\Delta m^2_{31}}{2}x_0\right)}{\frac{\Delta m^2_{31}}{2}(\delta x)}. \tag{1.20}$$

From $\Delta_{31} \gg 1$, we have $\frac{\Delta m^2_{31}}{2}x_1 \gg 1$ and $\frac{\Delta m^2_{31}}{2}x_0 \gg 1$, so that the sine function oscillates very fast, and the numerator is at most of order 1. As long as we are averaging over an interval δx that is not too small, the denominator is always large, such that $\langle\cos\Delta_{31}\rangle \overset{\delta x\to\infty}{\longrightarrow} 0$. The same calculation for $\langle\cos\Delta_{32}\rangle$ will also give a δx in the denominator with a numerator that is at most $\mathcal{O}(1)$, so it too goes to zero. Note that the exponential damping factors $e^{-\gamma_{ij}}$ only act to make this approach zero faster.

Averaging over these fast oscillations, we have

$$P(\nu_e \to \nu_e) = |U_{e1}|^4 e^{-\Gamma_1 t} + |U_{e2}|^4 e^{-\Gamma_2 t} + |U_{e3}|^4 e^{-\Gamma_3 t} + 2|U_{e1}|^2 |U_{e2}|^2 e^{-\gamma_{21}} \cos \Delta_{21}. \quad (1.21)$$

Using the PMNS matrix elements, we have

$$|U_{e1}|^2 = c_{12}^2 c_{13}^2, \quad |U_{e2}|^2 = s_{12}^2 c_{13}^2, \quad |U_{e3}|^2 = s_{13}^2, \quad (1.22)$$

so that

$$P(\nu_e \to \nu_e) = c_{13}^4 \left(c_{12}^4 e^{-\Gamma_1 t} + s_{12}^4 e^{-\Gamma_2 t} \right) + s_{13}^4 e^{-\Gamma_3 t} + 2 c_{12}^2 s_{12}^2 c_{13}^4 e^{-\gamma_{21}} \cos \Delta_{21}. \quad (1.23)$$

Given that θ_{13} is small, we can drop terms of $\mathcal{O}(\sin^4 \theta_{13})$, so that $c_{13}^4 \approx 1 - 2s_{13}^2$ and $s_{13}^4 \approx 0$. Expanding to first order in s_{13}^2, we have

$$P(\nu_e \to \nu_e) \approx (1 - 2s_{13}^2) \left(c_{12}^4 e^{-\Gamma_1 t} + s_{12}^4 e^{-\Gamma_2 t} + 2 c_{12}^2 s_{12}^2 e^{-\gamma_{21}} \cos \Delta_{21} \right) \quad (1.24)$$

Note that there are two types of terms in this expression: an oscillatory term (containing the time-dependent cosine $\cos \Delta_{21}$) and non-oscillatory terms (the rest). The oscillatory term is proportional to $e^{-\gamma_{21}} = e^{-(\Gamma_1 + \Gamma_2)t/2}$. Therefore, it is only important if $\Gamma_1 t$ and $\Gamma_2 t$ are not too large, so that $e^{-\gamma_{21}}$ is not too small.

Note that because of the averaging of the fast oscillations and due to the smallness of θ_{13}, there is no dependence on Γ_3. We have effectively reduced this to a two-neutrino system involving only ν_1 and ν_2.

Question 4. The fact that KamLAND observed oscillations means that the oscillatory term in $P(\nu_e \to \nu_e)$ must be non-negligible. This requires that $\Gamma_1 t$ and $\Gamma_2 t$ are not too large, so that $e^{-\gamma_{21}}$ is not too small. A reasonable requirement is that $\gamma_{21} = (\Gamma_1 + \Gamma_2)t/2 \ll 1$, which implies that $\Gamma_1 t \ll 1$ and $\Gamma_2 t \ll 1$. Using $\Gamma_i = \frac{m_i}{E_\nu} \frac{1}{\tau_i}$, we have

$$\Gamma_i t = \frac{m_i}{E_\nu} \frac{1}{\tau_i} t \approx \frac{m_i}{E_\nu} \frac{1}{\tau_i} \frac{L}{c} \ll 1 \implies \tau_i \gg \frac{m_i}{E_\nu} \frac{L}{c}. \quad (1.25)$$

Using $m_1 \sim 0.01$ eV, we find

$$m_2 = \sqrt{\Delta m_{21}^2 + m_1^2} \approx 0.0132 \text{ eV} = 1.32 \times m_1. \quad (1.26)$$

For $L = 180$ km, we have $L/c = 0.6$ ms. With $E_\nu = 4$ MeV, we have

$$\tau_1 \gg \frac{0.01 \text{ eV}}{4 \times 10^6 \text{ eV}} \times 0.6 \text{ ms} = 1.5 \times 10^{-12} \text{ s}, \quad \tau_2 \gg 1.3 \times \tau_1 \approx 2.0 \times 10^{-12} \text{ s}. \quad (1.27)$$

The lifetime of ν_1 and ν_2 must therefore be much larger than about 10^{-12} seconds or a picosecond. This is not such a strong limit, given that the muon lifetime is about 2.2×10^{-6} seconds, which is about a million times longer. The reason is that neutrinos are extremely boosted $\gamma = E_\nu / m_i \sim 4 \times 10^8$ in this case, so that their lifetime in the lab frame is much longer than in their rest frame.

Much better limits on the neutrino lifetime can be obtained by looking at astrophysical neutrinos, such as those from the Sun or from supernova 1987A.

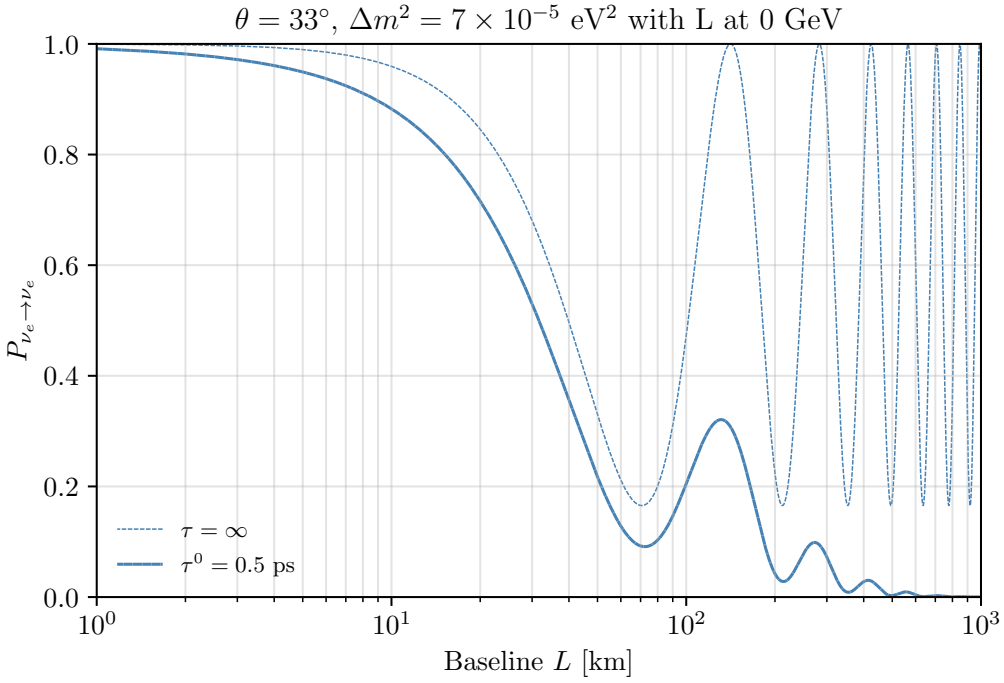

Figure 1: The ν_e survival probability $P(\nu_e \to \nu_e)$ as a function of the distance L in a 2-neutrino oscillation approximation with and without decay. As expected, the disappearance oscillation pattern is damped if the neutrinos decay. We take $E_\nu = 4$ MeV, $\Delta m^2_{21} = 7.5 \times 10^{-5}$ eV^2, $\theta_{12} = 33°$, and assume the rest-frame lifetimes of $\tau^0_1 \simeq \tau^0_2 = 10^{-12}$ s (solid blue).

You may ask what is the expectation for the neutrino lifetime in the Standard Model. It turns out *extremely* long. In principle, decays such as $\nu_2 \to \nu_1 + \nu_1 + \bar{\nu}_1$ are possible, but they are suppressed by the tiny neutrino mass (as well as by the tiny mass *splittings*). An extremely generous and naive estimate would be

$$\Gamma(\nu_2 \to \nu_1 + \nu_1 + \bar{\nu}_1) \sim \frac{G_F^2 m_2^5}{192\pi^3} \implies \frac{\Gamma_\nu}{\Gamma_\mu} \sim \left(\frac{m_2}{m_\mu}\right)^5 \sim 10^{-50}, \tag{1.28}$$

which gives a lifetime of $\tau_2 \sim 10^{44}$ seconds, which is about 10^{27} times the age of the Universe! In reality, the decay rate for this channel is even smaller because Weak neutral-currents couple diagonally in the mass basis, so that the process has to proceed though higher-order charged-current diagrams (another example of the so-called Glashow-Iliopoulos-Maiani (GIM) mechanism). So, if we see neutrino decay, it is an unequivocal sign of even more physics beyond the Standard Model (in addition to the neutrino masses themselves).

Question 5. If we interpret Γ_i as the absorption rate of neutrinos in the Earth, we can write $\Gamma_i = \sigma_i n_{\text{nucleons}}$, where σ_i is the absorption cross section of ν_i on nucleons and

n_{nucleons} is the number density of nucleons in the Earth. Using the same requirement as above, $\Gamma_i t \ll 1$, we have

$$\Gamma_i t = \sigma_i n_{\text{nucleons}} L \approx \sigma_i n_{\text{nucleons}} L \ll 1 \implies \sigma_i \ll \frac{1}{n_{\text{nucleons}} L}, \tag{1.29}$$

where we took $c = 1$ throughout. The number density of nucleons in the shallow crust of the Earth is about

$$n_{\text{nucleons}} = \frac{\rho_{\text{Earth}}}{m_{\text{nucleon}}} \approx \frac{2 \text{ g/cm}^3}{1.67 \times 10^{-24} \text{ g}} \approx 10^{24} \text{ cm}^{-3}. \tag{1.30}$$

Using $L = 180$ km[1], we have

$$\sigma_i \ll \frac{1}{10^{24} \text{ cm}^{-3} \times 1.8 \times 10^7 \text{ cm}} \approx 5.5 \times 10^{-32} \text{ cm}^2. \tag{1.31}$$

This limit is mostly independent of what neutrino mass state we are considering, as long as the neutrino mass is much smaller than its energy.

For comparison, the charged-current cross section of neutrinos on nucleons at $E_\nu = 4$ MeV is about $\sigma_{\nu N} \sim 10^{-42}$ cm^2, which is about 10 orders of magnitude smaller than the limit we just derived. This means that the absorption of neutrinos in the Earth is completely negligible at these energies, as expected.

Bonus Question 6. The CP asymmetry is defined as

$$A_{e\mu} = P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e). \tag{1.32}$$

The oscillation probability for antineutrinos can be obtained from Eq. (1.3) by replacing $U \to U^*$, so that $A^{ij}_{\alpha\beta} \to (A^{ij}_{\alpha\beta})^*$. Therefore, only the imaginary part of $A^{ij}_{\alpha\beta}$ changes sign, so that

$$A_{e\mu} = 4 \sum_{i>j} e^{-\gamma_{ij}} \operatorname{Im}(A^{ij}_{e\mu}) \sin \Delta_{ij}. \tag{1.33}$$

In the limit $\Delta_{21} \to 0$, we have $\sin \Delta_{21} \approx 0$, so that the only contribution comes from $i = 3$ and $j = 1, 2$:

$$A_{e\mu} = 4e^{-\gamma_{31}} \operatorname{Im}(A^{31}_{e\mu}) \sin \Delta_{31} + 4e^{-\gamma_{32}} \operatorname{Im}(A^{32}_{e\mu}) \sin \Delta_{32}. \tag{1.34}$$

Since $\Delta_{21} \to 0$, we have $\Delta_{32} \approx \Delta_{31}$, so that

$$A_{e\mu} \approx 4 \left(e^{-\gamma_{31}} \operatorname{Im}(A^{31}_{e\mu}) + e^{-\gamma_{32}} \operatorname{Im}(A^{32}_{e\mu})\right) \sin \Delta_{31}. \tag{1.35}$$

[1] Okay, this was my bad, many of you used the diameter of the Earth. Given the ambiguous phrasing of the question, this was perfectly fine, and gives a constraint that is about 2 orders of magnitude stronger. However, I should note that at energies of $E_\nu \sim 4$ MeV, we do not observe neutrino *oscillations* with $L = L_{\text{Earth}} \simeq 1.3 \times 10^4$ km. We do, however, observe oscillations with Earth's diameter at higher energies, so your limit would apply to cross sections for E_ν above the GeV energy scale.

Using the PMNS matrix elements, we have

$$\begin{aligned}
\mathrm{Im}(A^{31}_{e\mu}) &= \mathrm{Im}(U^*_{e3}U_{\mu 3}U_{e1}U^*_{\mu 1}) && (1.36)\\
&= s_{23}s_{13}c^2_{13}c_{12}\,\mathrm{Im}(e^{i\delta_{\mathrm{CP}}}(-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta_{\mathrm{CP}}})), && (1.37)\\
&= -c_{12}s_{12}c_{23}s_{23}c^2_{13}s_{13}\sin\delta_{\mathrm{CP}} = -J, && (1.38)\\
\mathrm{Im}(A^{32}_{e\mu}) &= \mathrm{Im}(U^*_{e3}U_{\mu 3}U_{e2}U^*_{\mu 2}) && (1.39)\\
&= s_{23}s_{13}c^2_{13}s_{12}\,\mathrm{Im}(e^{i\delta_{\mathrm{CP}}}(c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta_{\mathrm{CP}}})), && (1.40)\\
&= c_{12}S_{12}c_{23}s_{23}c^2_{13}s_{13}\sin\delta_{\mathrm{CP}} = +J. && (1.41)
\end{aligned}$$

No surprises here! We recover the Jarlskog invariant $J = c_{12}s_{12}c_{23}s_{23}c^2_{13}s_{13}$ that quantifies CP violation in the lepton sector. We had to calculate this explicitly just to get the signs right. Therefore,

$$A_{e\mu} \approx 4J\left(e^{-\gamma_{32}} - e^{-\gamma_{31}}\right)\sin\Delta_{31}. \tag{1.42}$$

This is interesting... First, note that if all $\Gamma_i = 0$, we recover $A_{e\mu} = 0$. This had to be the case, since CP violation in vacuum requires all three neutrino mass states to be involved and setting $\Delta_{21} \to 0$ effectively reduces the problem to a two-neutrino oscillation one. However, as long as $\Gamma_3 t$ is not too large, CP violation can still be manifested when $\Gamma_1 \neq \Gamma_2$ (so that $e^{-\gamma_{32}} \neq e^{-\gamma_{31}}$). This is because the criterion $\Gamma_1 \neq \Gamma_2$ distinguishes between ν_1 and ν_2, and the problem remains a three-neutrino oscillation one.