Solutions and discussion for Assignment 4b.

Question 1. By definition, the physical distance between us and a distant galaxy is given by $r(t) = a(t)r$, where r is the comoving distance (constant in time). Taking the time derivative, we find

$$v = \dot{r}(t) = \dot{a}(t)r + a(t)\dot{r} \tag{1.7}$$

where the last term is the peculiar velocity of the galaxy (its velocity relative to us in comoving coordinates , $v_{\text{pec}} = a\dot{r}$). Assuming that the galaxy is static in comoving coordinates (i.e., $\dot{r} = 0$), we have

$$v = \dot{a}(t)r = \frac{\dot{a}}{a}a(t)r = Hr(t). \tag{1.8}$$

This gives us Hubble's law. Note that this Hubble parameter H is time-dependent in general, but at the present time it is called the Hubble constant $H_0 = H(t_0) \simeq 70\,\text{km/s/Mpc}$.

Hubble's law implies that more distant galaxies are receding from us faster, which is a direct consequence of the expansion of the Universe. However, what is even more spectacular is that for sufficiently early times (very large distances), the Hubble parameter is itself varying with time, meaning that the expansion of the Universe is accelerating or decelerating depending on the energy content of the Universe (as we will see in Question 4).

The redshift z of a distant galaxy is defined as

$$1 + z = \frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}}, \tag{1.9}$$

where λ_{emitted} is the wavelength of light when it was emitted by the galaxy, and $\lambda_{\text{observed}}$ is the wavelength we observe today. The redshift is related to the scale factor by

$$1 + z = \frac{a(t_0)}{a(t_e)}, \tag{1.10}$$

meaning that the wavelength of light stretches with the expansion of the Universe. If a photon is emitted at time t_e when the scale factor is $a(t_e)$, and observed at time t_0 when the scale factor is $a(t_0)$, then the wavelength of the photon will have stretched by the same factor as the scale factor. Thus, we have

$$\frac{\lambda_{\text{observed}}}{\lambda_{\text{emitted}}} = \frac{a(t_0)}{a(t_e)}, \tag{1.11}$$

which is the stated relation for the redshift.

Question 2. Starting from the first Friedmann equation,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{1.12}$$

we take the time derivative of both sides to obtain

$$2H\dot{H} = \frac{8\pi G}{3}\dot{\rho} + \frac{2k}{a^3}\dot{a} = \frac{8\pi G}{3}\dot{\rho} + 2H\frac{k}{a^2}. \tag{1.13}$$

With that, we can write

$$\dot{H} = \frac{4\pi G}{3H}\dot{\rho} + \frac{k}{a^2}. \tag{1.14}$$

Now, using the second Friedmann equation expressed in terms of $\dot{H}$,

$$\dot{H} = -4\pi G(\rho + p) + \frac{k}{a^2}, \tag{1.15}$$

we can equate the two expressions for $\dot{H}$ to find

$$\frac{4\pi G}{3H}\dot{\rho} + \frac{k}{a^2} = -4\pi G(\rho + p) + \frac{k}{a^2}. \tag{1.16}$$

Simplifying, we finally find the desired continuity equation

$$\boxed{\dot{\rho} + 3H(\rho + p) = 0.} \tag{1.17}$$

Question 3. Starting from the continuity equation, we make use of $p = w\rho$ to find

$$\dot{\rho} + 3H\rho(1 + w) = 0 \quad \Longrightarrow \quad \frac{\dot{\rho}}{\rho} = -3H(1 + w) = -3(1 + w)\frac{\dot{a}}{a}. \tag{1.18}$$

where we used the definition of the Hubble parameter $H = \frac{\dot{a}}{a}$. Let us rewrite it using $d(\ln \rho)/dt = \dot{\rho}/\rho$ and $d(\ln a)/dt = \dot{a}/a$:

$$\int \frac{d\ln\rho}{dt}dt = -3(1 + w)\int \frac{d\ln a}{dt}dt. \tag{1.19}$$

We integrate this to find

$$\ln\rho - \ln\rho_0 = -3(1 + w)\left(\ln a - \ln a_0\right), \tag{1.20}$$

where ρ_0 and a_0 are the energy density and scale factor at some initial time t_0. Exponentiating both sides, we find

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}. \tag{1.21}$$

Thus, the energy density ρ scales with the scale factor $a(t)$ as

$$\boxed{\rho \propto a^{-3(1+w)} \quad \text{hence} \quad \lambda = 3(1 + w).} \tag{1.22}$$

Question 4. Starting from the first Friedmann equation for a flat Universe ($k = 0$) with no cosmological constant ($\Lambda = 0$),

$$H^2 = \frac{8\pi G}{3}\rho, \tag{1.23}$$

and substituting the result from Question 3, $\rho = \rho_0 a^{-3(1+w)}$, we find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)}. \tag{1.24}$$

Rearranging terms, we have

$$\dot{a} = \sqrt{\frac{8\pi G}{3}\rho_0} a^{-\frac{1}{2}(1+3w)}. \tag{1.25}$$

For the moment, let us assume that $w \neq -1$ and define $\tau = \frac{1}{2}(1+3w)$. In that case, we can write

$$\dot{a}a^{\tau} = \frac{1}{\tau+1}\frac{d}{dt}a^{\tau+1} = \sqrt{\frac{8\pi G}{3}\rho_0}. \tag{1.26}$$

to get

$$\frac{1}{\tau+1}\frac{d}{dt}a^{\tau+1} = \sqrt{\frac{8\pi G}{3}\rho_0}. \tag{1.27}$$

Integrating both sides from some reference time t_0, we get

$$\frac{1}{\tau+1}\left(a^{\tau+1}(t) - a^{\tau+1}(t_0)\right) = \sqrt{\frac{8\pi G}{3}\rho_0}(t-t_0). \tag{1.28}$$

Setting $a(t_0) = 0$, we solve for $a(t)$ to find

$$a(t) = \left(\frac{3(1+w)}{2}\sqrt{\frac{8\pi G}{3}\rho_0}t\right)^{\frac{2}{3(1+w)}}. \tag{1.29}$$

Now, we can evaluate this expression for the specific cases:

(a) For a radiation-dominated Universe ($w = 1/3$),

$$a(t) = \left(2\sqrt{\frac{8\pi G}{3}\rho_0}t\right)^{\frac{1}{2}} \propto t^{\frac{1}{2}}. \tag{1.30}$$

(b) For a matter-dominated Universe ($w = 0$),

$$a(t) = \left(\frac{3}{2}\sqrt{\frac{8\pi G}{3}\rho_0}t\right)^{\frac{2}{3}} \propto t^{\frac{2}{3}}. \tag{1.31}$$

(c) For a Universe dominated by a negative pressure fluid with $w = -1$ (cosmological constant), the exponent is undefined (division by zero), so we have to go back to our integration step in Eq. (1.27). In that case, we have,

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho_0}, \tag{1.32}$$

which we can integrate in $\ln a$ to find

$$\int d(\ln a) = \sqrt{\frac{8\pi G}{3}\rho_0}\int dt \quad \Longrightarrow \quad a(t) = a_0 e^{\sqrt{\frac{8\pi G}{3}\rho_0}t}, \tag{1.33}$$

where a_0 is the scale factor at some reference time t_0.

Alternatively, we can consider a Universe with just a cosmological constant, $\Lambda > 0$, and nothing else, $\rho = 0$. In that case, we go back to the first Friedmann equation, which reduces to the much simpler form:

$$H^2 = \frac{\Lambda}{3}, \tag{1.34}$$

since ρ is constant for $w = -1$. Taking the square root, we find $H = \sqrt{\frac{\Lambda}{3}}$ (Hubble is a constant for a dark energy dominated Universe!). Using the definition of the Hubble parameter, we have

$$\frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}}. \tag{1.35}$$

As before, we use $d(\ln a)/dt = \dot{a}/a$ and integrate from some reference time t_0 to get:

$$\int d(\ln a) = \sqrt{\frac{\Lambda}{3}} \int dt \quad \Longrightarrow \quad a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}} t}. \tag{1.36}$$

where a_0 is the scale factor at t_0. In this case, the scale factor grows exponentially with time. This is the mathematical description of the accelerated expansion of the Universe due to dark energy ($\Lambda > 0$).

All of the above should feel rather spectacular: by knowing what is inside the Universe (i.e., the equation of state parameter w), we can predict how fast it expands with time. And conversely, *by measuring how fast the Universe is expanding (i.e., measuring Hubble at different epochs), we can infer what is inside it!*

Bonus Question 5. This questions forces us to think about a multi-component Universe and introduces the concept of a critical density. Let us make use of this concept to define some standard parameters used in cosmology.

The first Friedmann equation for a homogeneous and isotropic Universe with no curvature and without a cosmological constant is:

$$H^2(a) = \frac{8\pi G}{3}\rho. \tag{1.37}$$

The critical density ρ_c is defined as the energy density required for a flat Universe ($k = 0$):

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{1.38}$$

We can define the density parameters for matter and radiation as

$$\Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_r = \frac{\rho_r}{\rho_c}. \tag{1.39}$$

The questions gives us that at present time, $a = 1$, we have $\Omega_{m,0} \approx 0.3$ and $\Omega_{r,0} \approx 5 \times 10^{-5}$ (the latter radiation density includes only contributions from photons. If you want to be more accurate, you can include neutrinos to find $\Omega_{r,0} \approx 9 \times 10^{-5}$).

With that, we can write the Friedmann equation in terms of the density parameters as

$$\boxed{H^2(a) = H_0^2 \left(\Omega_{m,0} a^{-3} + \Omega_{r,0} a^{-4}\right),} \tag{1.40}$$

where we used that $\rho_m \propto a^{-3}$ and $\rho_r \propto a^{-4}$ from Question 3 and H_0 is the Hubble constant today,

$$H_0 = H(a=1) = \sqrt{\frac{8\pi G}{3}(\rho_{m,0} + \rho_{r,0})}. \tag{1.41}$$

In this form, it becomes even more evident how Friedmann's equation is just telling us how the expansion rate of the Universe $H(a)$ changes with the scale factor a given its matter and radiation content.

Why is the critical density equivalent to a flat Universe? Write the first Friedmann equation as

$$H^2 = \frac{8\pi G}{3}\rho + \frac{k}{a^2} = H^2\Omega + \frac{k}{a^2} \implies \frac{k}{a^2} = H^2(1-\Omega), \tag{1.42}$$

where we defined the total density parameter $\Omega = \rho/\rho_c$. From this expression, we see that if $\Omega = 1$, then $k = 0$, which corresponds to a flat Universe.

To find the time of matter-radiation equality $t_{\rm eq}$, we set $\rho_m(t_{\rm eq}) = \rho_r(t_{\rm eq})$. Using the scaling relations from Question 3, we have

$$\rho_{m,0} a_{\rm eq}^{-3} = \rho_{r,0} a_{\rm eq}^{-4} \quad \implies \quad a_{\rm eq} = \frac{\rho_{r,0}}{\rho_{m,0}} = \frac{\Omega_{r,0}}{\Omega_{m,0}} = \frac{5\times 10^{-5}}{0.3} \approx 1.67 \times 10^{-4}. \tag{1.43}$$

This gives us the scale factor at matter-radiation equality. Given that the redshift is defined as $1 + z = \frac{a(t_0)}{a(t)}$, we can find the redshift at matter-radiation equality as

$$z_{\rm eq} = \frac{1}{a_{\rm eq}} - 1 \approx 6000. \tag{1.44}$$

Had we used the more accurate value for radiation density $\Omega_{r,0} = 9 \times 10^{-5}$ including neutrinos, we would have found $z_{\rm eq} \approx 3400$, which is the commonly quoted value in the literature.

To find the time $t_{\rm eq}$, we can integrate the Friedmann equation to find $a(t)$. Rearranging the Friedmann equation, we have

$$\frac{da}{dt} = H_0\sqrt{\Omega_{m,0} a^{-1} + \Omega_{r,0} a^{-2}} \implies dt = \frac{da}{H_0\sqrt{\Omega_{m,0} a^{-1} + \Omega_{r,0} a^{-2}}}. \tag{1.45}$$

Hence, we integrate from the singularity at $a = 0$ to $a_{\rm eq}$ to find $t_{\rm eq}$:

$$t(a_{\rm eq}) = \int_{a_\infty}^{a_{\rm eq}} \frac{da}{H_0\sqrt{\Omega_{m,0} a^{-1} + \Omega_{r,0} a^{-2}}} = \frac{1}{H_0\Omega_{m,0}} \int_0^{a_{\rm eq}} \frac{a\,da}{\sqrt{a + \frac{\Omega_{r,0}}{\Omega_{m,0}}}}. \tag{1.46}$$

By virtue of Eq. (1.43), we further simplify to

$$t(a_{\rm eq}) = \frac{1}{H_0\Omega_{m,0}} \int_0^{a_{\rm eq}} \frac{a\,da}{\sqrt{a + a_{\rm eq}}}. \tag{1.47}$$

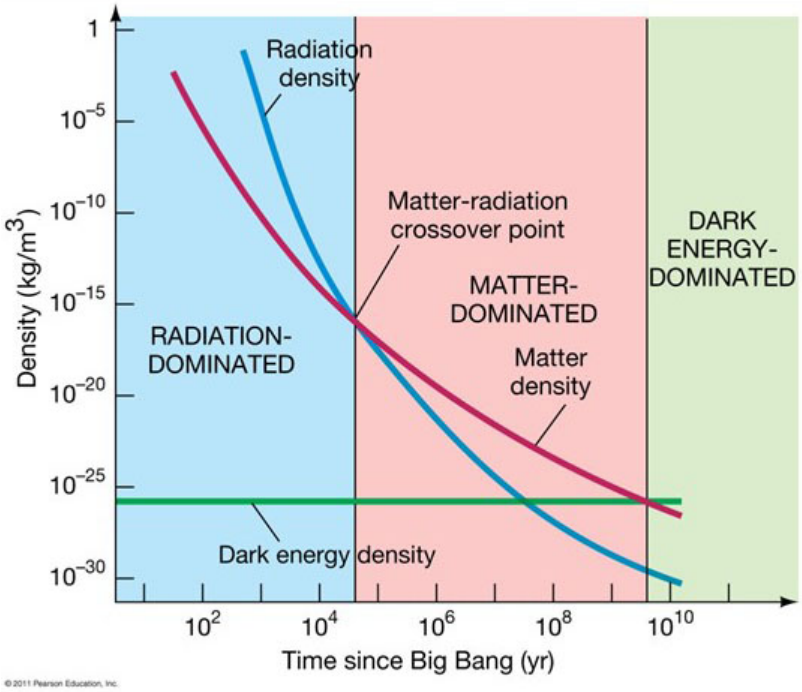

Figure 1: A schematic representation of the different epochs in the history of the Universe, highlighting the transitions between radiation domination, matter domination, and dark energy domination.

Solving this integral explicitly, we find the horrendous expression (sorry!!):

$$t_{\rm eq} = \frac{1}{H_0\Omega_{m,0}}\left[\frac{2}{3}\left(a - 2a_{\rm eq}\right)\sqrt{a + a_{\rm eq}}\right]_0^{a_{\rm eq}} = \frac{2a_{\rm eq}^{3/2}}{3H_0\Omega_{m,0}}\left[2 - \sqrt{2}\right] \tag{1.48}$$

$$\approx \frac{0.39\, a_{\rm eq}^{3/2}}{H_0\Omega_{m,0}} \simeq 39,000 \text{ years}, \tag{1.49}$$

where we used $H_0 = 70\,\mathrm{km/s/Mpc} \approx 2.27 \times 10^{-18}\,\mathrm{s}^{-1}$, $\Omega_{m,0} = 0.3$, $\Omega_{r,0} = 5 \times 10^{-5}$, and $a_{\rm eq} \approx 1.67 \times 10^{-4}$, we can compute $t_{\rm eq}$.

Thus, matter-radiation equality occurred when the universe was several tens of thousands of years old (at a redshift of about 6000). This is still very early compared to its current age of approximately 13.8 billion years, so the transition from radiation to matter domination happened quite early in the Universe's history.

We can ask what temperature this corresponds to. Using a relation between scale factor and temperature that we will derive in class, $a \propto 1/T$, we can estimate the temperature at matter-radiation equality. Given that the current temperature of the Cosmic Microwave Background (CMB) is approximately $T_0 \approx 2.725\,\mathrm{K}$, we can find the temperature at matter-radiation equality as

$$T_{\mathrm{eq}} = \frac{T_0}{a_{\mathrm{eq}}} \approx \frac{2.725\,\mathrm{K}}{1.67 \times 10^{-4}} \approx 16,300\,\mathrm{K} = 1.4\,\mathrm{eV}. \tag{1.50}$$

This temperature is significantly higher than the current CMB temperature, reflecting the hotter conditions of the early Universe at the time of matter-radiation equality. There is still *a lot* of interesting physics to happen in the Universe as it cools down from these temperatures, including the formation of neutral atoms (recombination), the decoupling of matter and radiation, and the eventual formation of the first stars and galaxies. However, it is not the earliest epoch we can study: before matter-radiation equality the Universe was even hotter and denser and processes such as the Big Bang Nucleosynthesis were happening. So, while matter-radiation equality is an important milestone in the history of the Universe, it is just one of many significant events that shaped the cosmos we observe today, and a relatively late one as far as particle physics is concerned!